Question 1: (Substitution Method)

(a) Determine $\int \frac{(\ln x)^2}{x}\,dx$. let $u = \ln x$, $du = \frac{1}{x}dx$

$= \int u^2\,du$

$= \frac{u^3}{3} + C$

$= \boxed{\frac{[\ln(x)]^3}{3} + C}$

[2]

(b) Determine $\int \frac{1+x}{1+x^2}\,dx$. $= \int \frac{1}{1+x^2}\,dx + \underbrace{\int \frac{x}{1+x^2}\,dx}_{\substack{u = 1+x^2 \\ du = 2x\,dx}}$

$= \boxed{\arctan(x) + \frac{1}{2}\ln|1+x^2| + C}$

[3]

(c) Evaluate $\int_1^2 \frac{e^{1/x}}{x^2}\,dx = I$

Let $u = \frac{1}{x}$, $du = \frac{-1}{x^2}dx$

$x = 1 \Rightarrow u = 1$

$x = 2 \Rightarrow u = \frac{1}{2}$

$\therefore I = -\int_1^{1/2} e^u\,du = \int_{\frac{1}{2}}^1 e^u\,du$

$= [e^u]_{1/2}^1$

$= \boxed{e - e^{1/2}}$

[2]

(d) Determine $\int_1^2 x\sqrt{x-1}\,dx$. $= I$

let $u = x-1$, so $x = u+1$

$du = dx$

$x = 1 \Rightarrow u = 0$

$x = 2 \Rightarrow u = 1$

$\therefore I = \int_0^1 (u+1)u^{1/2}\,du$

$= \int_0^1 u^{3/2} + u^{1/2}\,du$

$= \frac{2}{5}[u^{5/2}]_0^1 + \frac{2}{3}[u^{3/2}]_0^1$

$= \frac{2}{5} + \frac{2}{3}$

$= \boxed{\frac{16}{15}}$

[3]

Question 2: (Integration by Parts)

(a) Determine $\int_0^{1/2} x\cos(\pi x)\,dx$. $= I$

$u = x \quad dv = \cos(\pi x)\,dx$

$du = dx \quad v = \dfrac{\sin(\pi x)}{\pi}$

$\therefore I = \int_0^{1/2} u\,dv$

$= [uv]_0^{1/2} - \int_0^{1/2} v\,du$

$= \left[\dfrac{x\sin(\pi x)}{\pi}\right]_0^{1/2} - \int_0^{1/2} \dfrac{\sin(\pi x)}{\pi}\,dx$

$= \dfrac{(1/2)\sin(\pi/2)}{\pi} - 0 + \left[\dfrac{\cos(\pi x)}{\pi^2}\right]_0^{1/2}$

$= \dfrac{1}{2\pi} + \cancel{\dfrac{\cos(\pi/2)}{\pi^2}}^{\,0} - \dfrac{\cos(0)}{\pi^2}$

$= \dfrac{1}{2\pi} - \dfrac{1}{\pi^2}$

$= \boxed{\dfrac{\pi - 2}{2\pi^2}}$

[5]

(b) Determine $\int (\ln x)^2\,dx$.

$u = (\ln x)^2 \quad dv = dx$

$du = \dfrac{2\ln(x)}{x} \quad v = x$

$\therefore I = \int u\,dv$

$= uv - \int v\,du$

$= (\ln x)^2 x - \int \cancel{x}\cdot\dfrac{2\ln(x)}{\cancel{x}}\,dx$

$= x(\ln x)^2 - 2\underbrace{\int \ln(x)\,dx}_{\substack{u = \ln(x) \quad dv = dx \\ du = \frac{1}{x}dx \quad v = x}}$

$= x(\ln x)^2 - 2\left[uv - \int v\,du\right]$

$= x(\ln x)^2 - 2\left[x\ln(x) - \int \cancel{x}\,\dfrac{1}{\cancel{x}}\,dx\right]$

$= x(\ln x)^2 - 2x\ln(x) + 2x + C$

[5]

Question 3: (Trigonometric Integral)

Determine $\int \tan^5 x\, dx$

$$= \int \tan^3 x \tan^2 x \, dx$$

$$= \int \tan^3 x (\sec^2 x - 1)\, dx$$

$$= \int \tan^3 x \sec^2 x \, dx - \int \tan^3 x \, dx$$

$$= \int \tan^3 x \sec^2 x \, dx - \int \tan x (\sec^2 x - 1)\, dx$$

$$= \underbrace{\int \tan^3 x \sec^2 x \, dx}_{u = \tan x,\ du = \sec^2 x\, dx} - \int \underbrace{\tan x \sec^2 x \, dx}_{u = \tan x,\ du = \sec^2 x\, dx} + \int \underbrace{\frac{\sin x}{\cos x}\, dx}_{u = \cos x,\ du = -\sin x\, dx}$$

$$= \boxed{\frac{\tan^4 x}{4} - \frac{\tan^2 x}{2} - \ln|\cos x| + C}$$

[5]

Question 4: (Integral with irreducible quadratic)

Determine $\int \frac{5}{x^2 + 6x + 13}\, dx$

Completing the square,

$$= 5 \int \frac{1}{(x+3)^2 + 2^2}\, dx$$

$$= \boxed{5 \cdot \frac{1}{2} \arctan\left(\frac{x+3}{2}\right) + C}$$ by Formula #18.

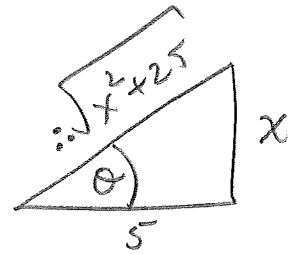
[5]

Question 5: (Trigonometric Substitution) Determine

$$I = \int \frac{1}{\sqrt{x^2+25}}\,dx$$

Let $x = 5\tan\theta$

$dx = 5\sec^2\theta\, d\theta$



$$\therefore I = \int \frac{5\sec^2\theta\, d\theta}{\sqrt{(5\tan\theta)^2+25}}$$

$$= \int \frac{5\sec^2\theta}{5\sec\theta}\, d\theta$$

$$= \int \sec\theta\, d\theta$$

$$= \ln\left|\sec\theta + \tan\theta\right| + C$$

$$= \boxed{\ln\left|\frac{\sqrt{x^2+25}}{5} + \frac{x}{5}\right| + C}$$

[10]

Question 6: (Partial Fractions) Determine

$$I = \int \frac{4x^2+3x+29}{(x+1)(x^2+9)}\,dx$$

$$\frac{4x^2+3x+29}{(x+1)(x^2+9)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+9}$$

$$= \frac{Ax^2+9A+Bx^2+Bx+Cx+C}{(x+1)(x^2+9)}$$

$$= \frac{(A+B)x^2+(B+C)x+9A+C}{(x+1)(x^2+9)}$$

$\therefore$ $A+B=4$ ① ① $\Rightarrow$ $B=4-A$

$B+C=3$ ② ② $\Rightarrow$ $C=3-B=3-(4-A)=A-1$

$9A+C=29$ ③ ③ $\Rightarrow$ $9A+(A-1)=29$

$\therefore 10A=30$

$\boxed{A=3}$, so $C=A-1$ and $B=4-A$

$\boxed{C=2}$ $\boxed{B=1}$

$$\therefore I = \int \frac{3}{x+1} + \frac{x+2}{x^2+9}\,dx$$

$$= 3\int \frac{1}{x+1}\,dx + \underbrace{\int \frac{x}{x^2+9}\,dx}_{u=x^2+9,\ du=2x\,dx} + 2\underbrace{\int \frac{1}{x^2+9}\,dx}_{\text{Formula \#18}}$$

$$= \boxed{3\ln|x+1| + \frac{1}{2}\ln|x^2+9| + \frac{2}{3}\arctan\left(\frac{x}{3}\right) + C}$$

[10]