Question 1: Acceleration due to gravity on the moon is a constant with a value of approximately $-8/5$ m/s^2. During the 1971 Apollo 15 moon mission, an astronaut standing on the moon's surface dropped a hammer and a feather (from rest) from the same height above the surface, and after $\sqrt{3/2}$ seconds both objects reached the surface at exactly the same time. From what height were the objects dropped?

$s''(t) = \frac{-8}{5}$, $s'(0)=0$, $s(\sqrt{\frac{3}{2}}) = 0$

$\Rightarrow s'(t) = -\frac{8}{5}t + C_1$,

$s'(0) = 0 \Rightarrow C_1 = 0$

$\therefore s'(t) = \frac{-8}{5}t$

$\Rightarrow s(t) = \left(\frac{-8}{5}\right)\frac{t^2}{2} + C_2$

$s(\sqrt{\frac{3}{2}}) = 0 \Rightarrow \left(\frac{-8}{5}\right)\left(\frac{1}{2}\right)\left(\sqrt{3/2}\right)^2 + C_2 = 0$

$\Rightarrow \left(\frac{-8}{5}\right)\left(\frac{1}{2}\right)\left(\frac{3}{2}\right) + C_2 = 0$

$\Rightarrow -\frac{6}{5} + C_2 = 0$

$\Rightarrow C_2 = +\frac{6}{5}$,

$\therefore s(t) = -\frac{4}{5}t^2 + \frac{6}{5}$

$\therefore s(0) = \boxed{\frac{6}{5} \text{ m}}$

[5]

Question 2: Find the most general antiderivative of each of the following:

(a) $f(x) = 2x(1 - x^{-3}) = 2x - 2x^{-2}$

$\therefore F(x) = 2\frac{x^2}{2} - 2\frac{x^{-1}}{-1} + C$

$\boxed{F(x) = x^2 + \frac{2}{x} + C}$

[2]

(b) $g(x) = \dfrac{5x - 3x^2\csc^2(x) + 1}{x^2} = \frac{5}{x} - 3\csc^2(x) + x^{-2}$

$\boxed{\therefore G(x) = 5\ln|x| + 3\cot(x) - \frac{1}{x} + C}$

[3]

Question 3: Use the definition of the definite integral in the form

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

to evaluate

$$\int_{-1}^{1} (4x - x^2)\,dx$$

Carefully set up the Riemann sum and clearly show the steps of your simplification.

$[a,b] = [-1,1]$

$\Delta x = \frac{b-a}{n} = \frac{1-(-1)}{n} = \frac{2}{n}$

$x_i = a + i\Delta x = -1 + i\frac{2}{n}$

$$f(x_i) = 4x_i - x_i^2 = 4\left(-1+\frac{2i}{n}\right) - \left(-1+\frac{2i}{n}\right)^2$$

$$= -4 + \frac{8i}{n} - 1 + \frac{4i}{n} - \frac{4i^2}{n^2}$$

$$= -5 + \frac{12i}{n} - \frac{4i^2}{n^2}$$

$$\int_{-1}^{1}(4x-x^2)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left(-5 + \frac{12i}{n} - \frac{4i^2}{n^2}\right)\left(\frac{2}{n}\right)$$

$$= \lim_{n\to\infty} \left[\left(\frac{-10}{n}\right)\left(\sum_{i=1}^{n} 1\right) + \left(\frac{24}{n^2}\right)\left(\sum_{i=1}^{n} i\right) - \left(\frac{8}{n^3}\right)\left(\sum_{i=1}^{n} i^2\right)\right]$$

$$= \lim_{n\to\infty} \left[\left(\frac{-10}{n}\right)(n) + \left(\frac{24}{n^2}\right)\left(\frac{n(n+1)}{2}\right) - \left(\frac{8}{n^3}\right)\left(\frac{n(n+1)(2n+1)}{6}\right)\right]$$

$$= \lim_{n\to\infty} \left[-10\cdot\cancel{\frac{n}{n}} + \frac{\overset{12}{\cancel{24}}}{\cancel{2}}\cdot\cancel{\frac{n}{n}}\cdot\underbrace{\frac{n+1}{n}}_{\to 1} - \frac{\overset{4}{\cancel{8}}}{\cancel{6}_3}\cdot\cancel{\frac{n}{n}}\cdot\underbrace{\frac{n+1}{n}}_{\to 1}\cdot\underbrace{\frac{2n+1}{n}}_{\to 2}\right]$$

$$= -10 + 12 - \frac{8}{3}$$

$$= 2 - \frac{8}{3}$$

$$= \boxed{-\frac{2}{3}}$$

Check: $\int_{-1}^{1}(4x-x^2)\,dx$

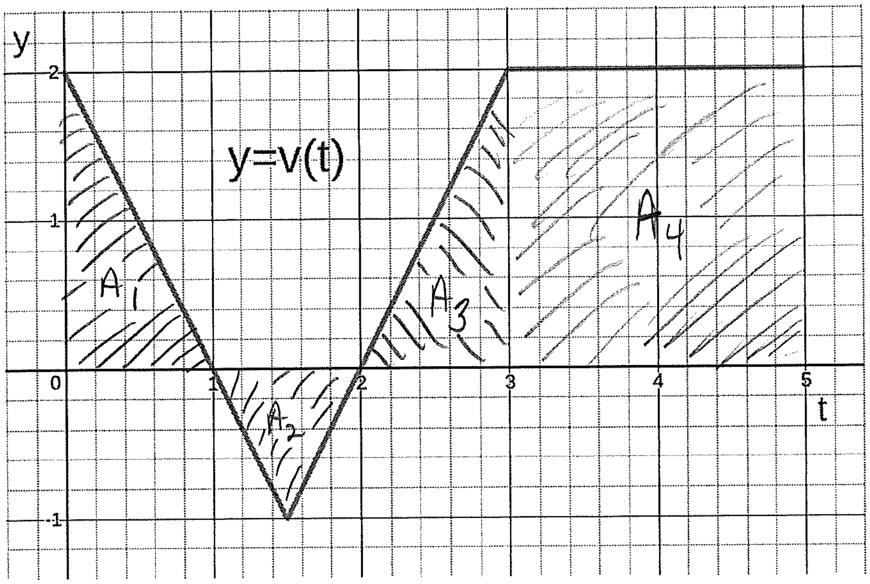
$= \frac{4}{2}[x^2]_{-1}^{1} - \frac{1}{3}[x^3]_{-1}^{1}$

$= 2[1-1] - \frac{1}{3}[1-(-1)]$

$= -\frac{2}{3}$ ✓

[10]

Question 4: The following is the graph of the velocity function for a particle moving along a straight line over the time interval $0 \le t \le 5$:



$A_1 = (\frac{1}{2})(1)(2) = 1$

$A_2 = (\frac{1}{2})(1)(1) = \frac{1}{2}$

$A_3 = (\frac{1}{2})(1)(2) = 1$

$A_4 = (2)(2) = 4$

(a) Write a definite integral representing the total displacement of the particle and determine its value.

$$\text{Displacement} = \int_0^5 v(t)\,dt = A_1 + A_3 + A_4 - A_2$$
$$= 1 + 1 + 4 - \frac{1}{2}$$
$$= \boxed{\frac{11}{2}}$$

[3]

(b) Write a definite integral representing the total <u>distance</u> travelled by the particle and determine its value.

$$\text{Distance} = \int_0^5 |v(t)|\,dt$$
$$= A_1 + A_2 + A_3 + A_4$$
$$= 1 + 1 + 4 + \frac{1}{2} = \boxed{\frac{13}{2}}$$

[4]

(c) Determine the average velocity of the object over the five second time interval.

$$v_{ave} = \frac{1}{5-0}\int_0^5 v(t)\,dt = \frac{1}{5}\left(\frac{11}{2}\right) = \boxed{\frac{11}{10}}$$

[3]

Question 5: Determine the following integrals:

(a) $\int_{-2}^{0} 2x + 5\,dx$ $= \left[\cancel{2}x^2 + 5x\right]_{-2}^{0}$ $= \left[0^2 + 5(0)\right] - \left[(-2)^2 + 5(-2)\right]$

$= \boxed{6}$

[3]

(b) $\int \frac{1 + x + x^2}{\sqrt{x}}\,dt$ $= \int x^{-\frac{1}{2}} + x^{\frac{1}{2}} + x^{3/2}\,dx$

$= \boxed{2x^{1/2} + \frac{2}{3}x^{3/2} + \frac{2}{5}x^{5/2} + C}$

[2]

(c) $\int_{-\pi/4}^{\pi/4} \sec(x)\tan(x)\,dx$ $= \left[\sec(x)\right]_{-\frac{\pi}{4}}^{\frac{\pi}{4}}$

$= \sec(\frac{\pi}{4}) - \sec(-\frac{\pi}{4})$

$= \sqrt{2} - \sqrt{2}$

$= \boxed{0}$

[3]

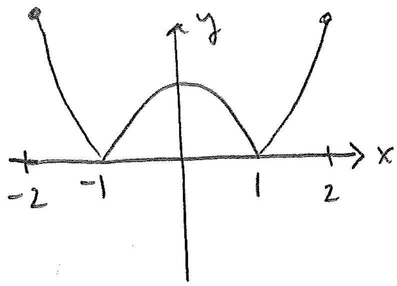
(d) $\int \frac{te^t - 1 - t}{t}\,dt$ $= \int e^t - \frac{1}{t} - 1\,dt$

$= \boxed{e^t - \ln|t| - t + C}$

[3]

Question 6: Compute $\int_{-2}^{2} |1-x^2|\, dx$ $= I$

$y = |1-x^2|$ has graph



Using graph,

$$I = 2\int_0^1 1-x^2\,dx + 2\int_1^2 -(1-x^2)\,dx$$

$$= 2\left[x-\frac{x^3}{3}\right]_0^1 + 2\left[\frac{x^3}{3}-x\right]_1^2$$

$$= 2\left[\left(1-\frac{1}{3}\right)-(0)\right] + 2\left[\left(\frac{8}{3}-2\right)-\left(\frac{1}{3}-1\right)\right]$$

$$= \frac{4}{3}+\frac{4}{3}+\frac{4}{3}$$

$$= \boxed{4}$$

[5]

Question 7: Let $f(x) = \int_0^{x^2} \frac{t}{1+t^2}\, dt$. Here the domain of f is all real numbers. Find all solutions to $f''(x) = 0$.

$$f'(x) = \frac{x^2}{1+(x^2)^2}\cdot 2x = \frac{2x^3}{1+x^4}$$

$$f''(x) = \frac{(1+x^4)(6x^2)-(2x^3)(4x^3)}{(1+x^4)^2} = \frac{6x^2+6x^6-8x^6}{(1+x^4)^2} = \frac{6x^2-2x^6}{(1+x^4)^4}$$

So $f''(x)=0 \Rightarrow 6x^2-2x^6=0$

$\Rightarrow 2x^2(3-x^4)=0$

$\Rightarrow \boxed{x=0,\ x=\pm 3^{1/4}}$

[5]