Question 1:

(a) Determine the linear approximation $T_1(x)$ for $f(x) = 2x^2 - \ln(x)$ at $a = 1$ and use it to approximate $f(1.1)$. State your answer as a single simplified fraction.

$f(x) = 2x^2 - \ln(x)$; $f(1) = 2(1)^2 - \cancel{\ln(1)}^{0} = 2$

$f'(x) = 4x - \frac{1}{x}$; $f'(1) = 4 - 1 = 3$

$\therefore T_1(x) = f(a) + f'(a)(x-a)$

$= \boxed{2 + 3(x-1)}$

$f(1.1) \approx T_1(1.1) = 2 + 3(1.1 - 1)$

$= 2 + 3\left(\frac{1}{10}\right)$

$= \frac{20+3}{10}$

$= \boxed{\frac{23}{10}}$

[5]

(b) Give an error bound on your approximation in part (a). Again, state your answer as a single simplified fraction.

$R_1(x) = \frac{f''(z)}{2}(x-a)^2$ where $x = 1.1$, $a = 1$, $1 < z < 1.1$

$f'(z) = 4z - \frac{1}{z} \Rightarrow f''(z) = 4 + \frac{1}{z^2}$

$\therefore |R_1(1.1)| = \left|\frac{1}{2}\left(4 + \frac{1}{z^2}\right)(1.1-1)^2\right|$

$< \frac{1}{2}\left(4 + \frac{1}{1^2}\right)\left(\frac{1}{10}\right)^2$

$= \frac{5}{(2)(100)}$

$= \boxed{\frac{1}{40}}$

[5]

Question 2:

(a) Use a Taylor polynomial of degree 2 to approximate $\sqrt{4.1}$. State your answer as a single simplified fraction.

$f(x) = x^{1/2}$, $a = 4$ $\Rightarrow$ $f(a) = f(4) = 2$

$f'(x) = \frac{1}{2}x^{-1/2}$; $f'(4) = \frac{1}{4}$

$f''(x) = -\frac{1}{4}x^{-3/2}$; $f''(4) = \frac{-1}{32}$.

$$T_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2$$
$$= 2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2$$

$$\therefore \sqrt{4.1} \approx T_2(4.1) = 2 + \frac{1}{4}(4.1-4) - \frac{1}{64}(4.1-4)^2$$
$$= 2 + \left(\frac{1}{4}\right)\left(\frac{1}{10}\right) - \left(\frac{1}{64}\right)\left(\frac{1}{100}\right)$$
$$= \frac{12800 + 160 - 1}{6400} = \boxed{\frac{12959}{6400}}$$

[5]

(b) Give an error bound on your approximation in part (a). Again, state your answer as a single simplified fraction.

$R_2(x) = \frac{f'''(z)}{3!}(x-a)^3$ where $x = 4.1$, $a = 4$, $4 < z < 4.1$.

$$|R_2(4.1)| = \left|\left(\frac{1}{3!}\right)\left(\frac{3}{8}\right)\left(\frac{1}{z^{5/2}}\right)(4.1-4)^3\right|$$

$$< \left|\left(\frac{1}{\cancel{3}\cdot 2\cdot 1}\right)\left(\frac{\cancel{3}}{2^3}\right)\left(\frac{1}{2^5}\right)\left(\frac{1}{10^3}\right)\right|$$

$$= \boxed{\frac{1}{2^9\, 10^3}}$$

[5]

Question 3:
Find the Taylor series about $a = -2$ for $f(x) = 1 + x + 2x^2 + 3x^3$. You should be able to write all terms of the series.

$$f(x) = 1 + x + 2x^2 + 3x^3 \ ; \quad f(-2) = 1 - 2 + 2(-2)^2 + 3(-2)^3 = -17$$

$$f'(x) = 1 + 4x + 9x^2 \ ; \quad f'(-2) = 1 + 4(-2) + 9(-2)^2 = 29$$

$$f''(x) = 4 + 18x \ ; \quad f''(-2) = 4 + 18(-2) = -32$$

$$f'''(x) = 18 \ ; \quad f'''(-2) = 18$$

$$f^{(k)}(x) = 0 \text{ for } k \geq 4.$$

$$\therefore T(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3$$

$$= \boxed{-17 + 29(x+2) - 16(x+2)^2 + 3(x+2)^3}$$

[5]

Question 4: Find the first four nonzero terms of the Taylor series about $a = 1$ for $g(x) = \dfrac{4}{2-3x}$ and state the open interval of convergence.

$$g(x) = \frac{4}{2-3x}$$

$$= \frac{4}{2-3(x-1)-3}$$

$$= \frac{4}{-1-3(x-1)}$$

$$= \frac{-4}{1}\left[\frac{1}{1-[-3(x-1)]}\right]$$

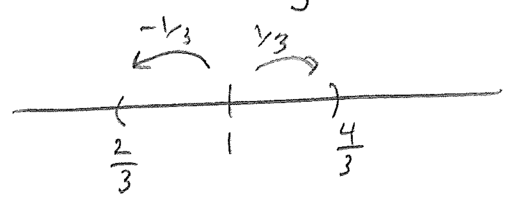
$$= -4\left[1 + [-3(x-1)] + [-3(x-1)]^2 + [-3(x-1)]^3 + \cdots\right]$$

$$= \boxed{-4 + 12(x-1) - 36(x-1)^2 + 108(x-1)^3 + \cdots}$$

Converges for

$$|-3(x-1)| < 1$$

$$\Rightarrow \quad |x-1| < \frac{1}{3}$$



$$\therefore I = \left(\frac{2}{3}, \frac{4}{3}\right)$$

[5]

Question 5: Find the Maclaurin polynomial of degree 11 for $f(x) = x^2 \arctan(2x^3)$.

$$\arctan(2x^3) = (2x^3) - \frac{(2x^3)^3}{3} + \frac{(2x^3)^5}{5} - \cdots$$

$$= 2x^3 - \frac{8}{3}x^9 + \frac{32}{5}x^{15} - \cdots$$

$$\therefore\ x^2\arctan(2x^3) = \underbrace{2x^5 - \frac{8}{3}x^{11}}_{T_{11}(x)} + \frac{32}{5}x^{17} - \cdots$$

$$\boxed{\therefore\ T_{11}(x) = 2x^5 - \frac{8}{3}x^{11}}$$

[3]

Question 6: Use series to find the limit: $\lim_{x\to 0} \dfrac{e^x - 1 - x}{\cos(x) - 1}$

$$= \lim_{x\to 0} \frac{\left(\cancel{1} + \cancel{x} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right) - \cancel{1} - \cancel{x}}{\left(\cancel{1} - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right) - \cancel{1}}$$

$$= \lim_{x\to 0} \frac{\cancel{x^2}\left(\frac{1}{2!} + \frac{x}{3!} + \frac{x^2}{4!} + \cdots\right)}{\cancel{x^2}\left(-\frac{1}{2!} + \frac{x^2}{4!} - \frac{x^4}{6!} + \cdots\right)}$$

$$= \boxed{-1}$$

[3]

Question 7: Find the first three non-zero terms of the Maclaurin series for $f(x) = e^{-x}\sin(x)$.

$$e^{-x}\sin(x) = \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots\right)\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)$$

e^{-x} \ $\sin(x)$	x	$-\frac{x^3}{6}$	$+\frac{x^5}{5!}$
1	x	$-\frac{x^3}{6}$	$+\frac{x^5}{5!}$
$-x$	$-x^2$	$+\frac{x^4}{6}$	$-\frac{x^6}{5!}$
$\frac{x^2}{2}$	$\frac{x^3}{2}$	$-\frac{x^5}{12}$	$+\frac{x^7}{2\cdot 5!}$
$-\frac{x^3}{6}$	$-\frac{x^4}{6}$	$+\frac{x^6}{36}$	$-\frac{x^8}{6\cdot 5!}$

$$\therefore\ T(x) = x - x^2 + \left(\frac{x^3}{2} - \frac{x^3}{6}\right) + \cdots$$

$$= \boxed{x - x^2 + \frac{1}{3}x^3 + \cdots}$$

[4]

Question 8: Find the radius of convergence R and open interval of convergence $\mathcal{I}$ for the power series

$$f(x) = \sum_{k=1}^{\infty} \frac{2^k (x+1)^{2k}}{k^2} \quad \Big\} \; a = -1$$

$$\lim_{k\to\infty} \left| \frac{u_{k+1}(x)}{u_k(x)} \right| < 1$$

$$\lim_{k\to\infty} \left| \frac{2^{k+1}(x+1)^{2(k+1)}}{(k+1)^2} \cdot \frac{k^2}{2^k (x+1)^{2k}} \right| < 1$$

$$\lim_{k\to\infty} \left| \frac{2^{\cancel{k}+1}}{\cancel{2^k}} \cdot \underbrace{\frac{k^2}{(k+1)^2}}_{\to 1} \cdot \frac{(x+1)^{2k+2}}{(x+1)^{2k}} \right| < 1$$

$$\therefore \left| 2(x+1)^2 \right| < 1$$

$$\therefore |x+1|^2 < \frac{1}{2}$$

$$|x-(-1)| < \frac{1}{\sqrt{2}}$$

$$\boxed{\therefore R = \frac{1}{\sqrt{2}}, \quad \mathcal{I} = \left(-1-\frac{1}{\sqrt{2}}, -1+\frac{1}{\sqrt{2}}\right).}$$

[5]

Question 9: Find the radius of convergence R and open interval of convergence $\mathcal{I}$ for the power series

$$f(x) = \sum_{k=1}^{\infty} \frac{(-1)^k k! (x-3)^k}{\sqrt{k}} \quad \Big\} \; a = 3$$

$$\lim_{k\to\infty} \left| \frac{u_{k+1}(x)}{u_k(x)} \right| < 1$$

$$\lim_{k\to\infty} \left| \frac{(-1)^{k+1}(k+1)!\,(x-3)^{k+1}}{\sqrt{k+1}} \cdot \frac{\sqrt{k}}{(-1)^k k!\,(x-3)^k} \right| < 1$$

$$\lim_{k\to\infty} \left| \underbrace{\frac{(-1)^{k+1}}{(-1)^k}}_{=1} \cdot \underbrace{\frac{(k+1)!}{k!}}_{=k+1} \cdot \underbrace{\frac{\sqrt{k}}{\sqrt{k+1}}}_{\to 1} \cdot \frac{(x-3)^{k+1}}{(x-3)^k} \right| < 1$$

$$\lim_{k\to\infty} \left| (k+1)(x-3) \right| < 1 \implies \infty < 1 \text{ unless } x = 3.$$

$$\boxed{\therefore R = 0, \; \mathcal{I} \text{ does not exist.}}$$

[5]