Question 1:

(a) Determine the coefficient of $1/(z+1)$ in the partial fraction decomposition of

$$\frac{z^3+4z+9}{(2z+2)(z-3)^5} = \frac{z^3+4z+9}{2(z+1)(z-3)^5}$$

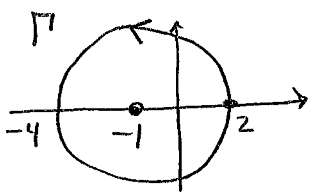
$$a = \lim_{z\to -1} \frac{\cancel{(z+1)}(z^3+4z+9)}{2\cancel{(z+1)}(z-3)^5}$$

$$= \frac{(-1)^3+4(-1)+9}{2(-1-3)^5}$$

$$= \frac{4}{2(-4)^5} = \boxed{-\frac{1}{2^9} \text{ or } \frac{-1}{512}}$$

[3]

(b) Compute $\int_\Gamma \frac{z^3+4z+9}{(2z+2)(z-3)^5}\,dz$ where Γ is the circle of radius 3 centred at $z=-1$. Explain your reasoning.



Using part (a), $\frac{z^3+4z+9}{(2z+2)(z-3)^5} = \frac{a}{z+1} + f(z)$

where $f(z)$ is analytic inside and on Γ. Since -1 is inside Γ, we have

$$\int_\Gamma \frac{a}{z+1} + f(z)\,dz = a\int_\Gamma \frac{1}{z+1}\,dz + \cancel{\int_\Gamma f(z)\,dz}^{\,0}$$

$$= a\cdot 2\pi i$$

$$= \left(-\frac{1}{512}\right)(2\pi i) = \boxed{-\frac{\pi i}{256}}$$

[2]

Question 2: Show that $\overline{\sin(z)} = \sin(\overline{z})$.

$$\overline{\sin(z)} = \overline{\left(\frac{e^{iz}-e^{-iz}}{2i}\right)}$$

$$= \overline{\left(\frac{e^{i(x+iy)}-e^{-i(x+iy)}}{2i}\right)}$$

$$= \overline{\left(\frac{e^{ix}e^{-y}-e^{-ix}e^{y}}{2i}\right)}$$

$$= \frac{e^{-ix}e^{-y}-e^{ix}e^{y}}{-2i}$$

$$= \frac{e^{i(x-iy)}-e^{-i(x-iy)}}{2i}$$

$$= \frac{e^{i\overline{z}}-e^{-i\overline{z}}}{2i}$$

$$= \sin(\overline{z})$$

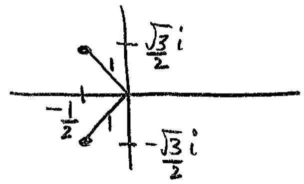
[5]

Question 3: Find all solutions to $e^{2z} + e^z + 1 = 0$

$$(e^z)^2 + (e^z) + 1 = 0$$

$$e^z = \frac{-1 \pm \sqrt{1^2 - 4(1)(1)}}{2}$$

$$= \frac{-1 \pm \sqrt{3}\, i}{2} \longrightarrow$$



$$= e^{i(2\pi/3)}, \; e^{i(4\pi/3)}$$

$$\therefore \quad z = \begin{cases} i\left(\frac{2\pi}{3}\right) + i2k\pi \\ i\left(\frac{4\pi}{3}\right) + i2k\pi \end{cases} = \begin{cases} i\,2\left(\frac{3k+1}{3}\right)\pi, \\ i\,2\left(\frac{3k+4}{3}\right)\pi \end{cases} \quad k \in \mathbb{Z}.$$

[5]

Question 4: Compute $(1+i)^{(1+i)}$ $= e^{(1+i)\log(1+i)}$

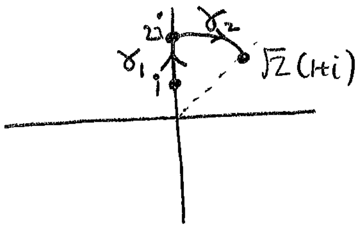
$$= e^{(1+i)\left[\mathrm{Log}\sqrt{2} + i\frac{\pi}{4} + i2k\pi\right]}$$

$$= e^{\mathrm{Log}\sqrt{2}} \; e^{-\frac{\pi}{4} - 2k\pi} \; e^{i\frac{\pi}{4} + i2k\pi + i\,\mathrm{Log}\sqrt{2}}$$

$$= \sqrt{2} \; e^{-(8k+1)\pi/4} \; e^{i\left[(8k+1)\pi/4 + \mathrm{Log}\sqrt{2}\right]}, \quad k \in \mathbb{Z}.$$

[5]

Question 5: Let Γ be the contour consisting of two smooth curves γ_1 and γ_2 as follows: γ_1 is the straight line segment from i to $2i$. γ_2 is the arc of the circle of centre $z = 0$ and radius 2 which begins at $2i$ and proceeds clockwise, ending at $\sqrt{2}(1+i)$. Using this contour evaluate



$$\int_\Gamma \frac{1}{(\overline{z})}\,dz$$

γ_1: $z(t) = it$, $t = 1..2$
$z'(t) = i$

$$\int_{\gamma_1} \frac{1}{\bar z}\,dz = \int_1^2 \frac{1}{-\cancel{i}t}\cdot \cancel{i}\,dt = -\ln|t|\Big|_1^2 = -\ln 2.$$

γ_2: $z(t) = 2e^{it}$, $t = \frac{\pi}{2}..\frac{\pi}{4}$
$z'(t) = 2ie^{it}$.

$$\int_{\gamma_1} \frac{1}{\bar z}\,dz = \int_{\frac{\pi}{2}}^{\frac{\pi}{4}} \frac{1}{\cancel{2}e^{-it}}\cdot \cancel{2}ie^{it}\,dt = \frac{i-(-1)}{2} = \frac{1+i}{2}$$

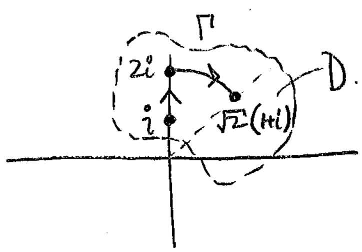
$$= \int_{\frac{\pi}{2}}^{\pi/4} i e^{2it}\,dt$$

$$= \frac{\cancel{i}}{2\cancel{i}}\Big[e^{2it}\Big]_{\frac{\pi}{2}}^{\pi/4}$$

$$\therefore \int_\Gamma \frac{1}{(\bar z)}\,dz = \boxed{-\ln 2 + \frac{i-1}{2}.}$$

[6]

Question 6: Using the same contour Γ as in the previous question, evaluate



$$\int_\Gamma \frac{1}{z}\,dz$$

$f(z)$ is continuous throughout D with antiderivative $\mathrm{Log}(z)$.

$$\therefore \int_\Gamma \frac{1}{z}\,dz = \mathrm{Log}(z)\Big|_i^{\sqrt{2}(1+i)}$$

$$= \mathrm{Log}(\sqrt{2}(1+i)) - \mathrm{Log}(i)$$

$$= \ln\sqrt{2} + i\frac{\pi}{4} - i\frac{\pi}{2}$$

$$= \ln\sqrt{2} - i\pi/4$$

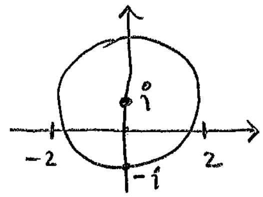
[4]

Question 7: Compute the following integrals. In each case the contour is traversed once in the positive direction. Make reference to any theorems used.

(a) $\displaystyle\int_\Gamma \frac{\sin(z)}{z^2-z-6}\,dz = \int_\Gamma \underbrace{\frac{\sin(z)}{(z+2)(z-3)}}_{f(z)}\,dz$

Here Γ is the circle of centre $z=i$ and radius 2.

Γ:



$f(z)$ analytic inside and on Γ,

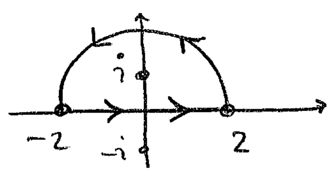
so $\displaystyle\int_\Gamma f(z)\,dz = \boxed{0}$ by Cauchy's Integral Thm.

[3]

(b) $\displaystyle\int_\Gamma \frac{e^z}{z^2+1}\,dz$

Here Γ is the top half of the circle of centre $z=0$ and radius 2 followed by the line segment from -2 to 2.

Γ:



$$\int_\Gamma \frac{e^z}{z^2+1}\,dz = \int_\Gamma \frac{e^z/(z+i)}{(z-i)}\,dz$$

by Cauchy's Integral Formula

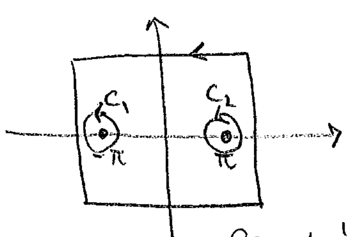
$$= 2\pi i \left[\frac{e^z}{z+i}\right]\Bigg|_{z=i} = \cancel{2}\pi \cancel{i}\,\frac{e^i}{\cancel{2}\cancel{i}} = \boxed{\pi e^i}$$

[3]

(c) $\displaystyle\int_\Gamma \frac{\cos^2(z)}{z^2-\pi^2}\,dz$

Here Γ is a square of area 64 and is such that each side is parallel to one coordinate axis and bisected by another.

Γ:



$\displaystyle\int_\Gamma = \int_{C_1} + \int_{C_2}$ by the deformation invariance Thm

$$= \int_{C_1} \frac{\cos^2(z)/(z-\pi)}{(z+\pi)}\,dz + \int_{C_2} \frac{\cos^2(z)/(z+\pi)}{(z-\pi)}\,dz$$

Cauchy's Integral Formula.

$$= \cancel{2}\cancel{\pi} i \left(\frac{\cos^2(-\pi)}{-\pi-\pi}\right) + \cancel{2}\cancel{\pi} i \left(\frac{\cos^2(\pi)}{\pi+\pi}\right)$$

$$= -i + i = \boxed{0}$$

[4]