Question 1: Simplify and express your answer in the form $a+ib$ where a and b are real:

$$\left[1+\frac{3}{(1+2i)}\right]^2$$

$$= \left(\frac{1+2i+3}{1+2i}\right)^2$$

$$= \left(\frac{4+2i}{1+2i}\cdot\frac{1-2i}{1-2i}\right)^2$$

$$= \left(\frac{8-6i}{5}\right)^2$$

$$= \frac{64-96i-36}{25}$$

$$= \boxed{\frac{28}{25}-\frac{96}{25}i}$$

[5]

Question 2: Express $z=\dfrac{i(1-i)(-\sqrt{3}+i)}{5}$ in form $z=re^{i\theta}$ where r and θ are real.

$$z = \frac{\left[e^{i\pi/2}\right]\left[\sqrt{2}\,e^{-i\pi/4}\right]\left[2e^{i\frac{5\pi}{6}}\right]}{5}$$

$$= \frac{2\sqrt{2}}{5}\,e^{i\left(\frac{\pi}{2}-\frac{\pi}{4}+\frac{5\pi}{6}\right)}$$

$$= \frac{2\sqrt{2}}{5}\,e^{i\left(\frac{6\pi-3\pi+10\pi}{12}\right)}$$

$$= \boxed{\frac{2\sqrt{2}}{5}\,e^{i\frac{13\pi}{12}}}$$

[5]

Question 3: Determine all values of $(-32-32i)^{1/5}$.

$$-32-32i = 32(-1-i) = 32\sqrt{2}\, e^{i\frac{5\pi}{4}}$$

$$\therefore (-32-32i)^{1/5} = 2\cdot 2^{\frac{1}{10}}\, e^{i\left(\frac{5\pi}{4}+2k\pi\right)/5}, \quad k = 0,1,2,3,4$$

$$= 2^{\frac{11}{10}} e^{i\frac{\pi}{4}},\ 2^{11/10} e^{i\,13\pi/20},$$

$$2^{\frac{11}{10}} e^{i\,21\pi/20},\ 2^{11/10} e^{i\frac{29\pi}{20}},$$

$$2^{\frac{11}{10}} e^{i\frac{37\pi}{20}}.$$

[5]

Question 4: Find all solutions to $(z+1)^5 = z^5$. (Hint: think about $\omega = 1^{1/5}$. You may leave your answer in terms of ω.)

$$(z+1)^5 = z^5$$

$$\Rightarrow \left(\frac{z+1}{z}\right)^5 = 1$$

$$\Rightarrow \frac{z+1}{z} = 1, \omega, \omega^2, \omega^3, \omega^4 \quad \text{where } \omega = e^{i\,2\pi/5}$$

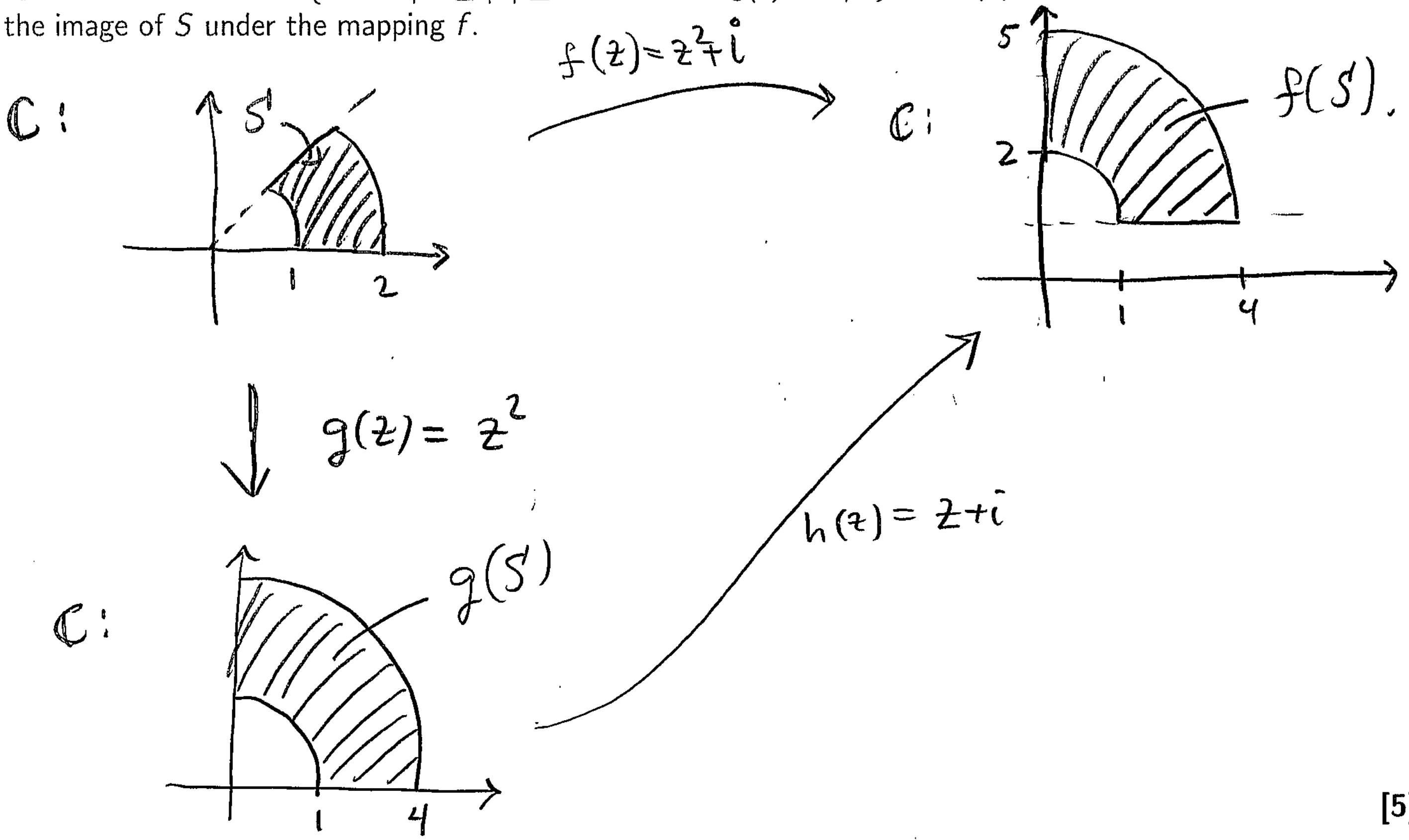
$$\Rightarrow 1 + \frac{1}{z} = 1, \omega, \omega^2, \omega^3, \omega^4$$

$$\Rightarrow \frac{1}{z} = \cancel{0}, \omega-1, \omega^2-1, \omega^3-1, \omega^4-1$$

$$\Rightarrow z = \frac{1}{\omega-1}, \frac{1}{\omega^2-1}, \frac{1}{\omega^3-1}, \frac{1}{\omega^4-1}$$

[5]

Question 5: Let $S = \{z \in \mathbb{C} \mid 1 \le |z| \le 2 \text{ and } 0 \le \mathrm{Arg(z)} \le \pi/4\}$ and $f(z) = z^2 + i$. Sketch S and $f(S)$, the image of S under the mapping f.



[5]

Question 6: Determine all points where the function $f(z) = x^2 - y^2 + 2x + 1 + 2ixy + i2y$ is analytic. Include all details in your conclusion.

$f(z) = (x^2-y^2+2x+1) + i\,2y(x+1)$

$= u+iv$ say.

$u_x = 2x+2 = v_y$ ✓

$u_y = -2y = -v_x$ ✓

$\therefore$ C.R. Equations are satisfied at every $z \in \mathbb{C}$ and all partial derivatives are continuous on $\mathbb{C}$, so f is entire.

[5]

Question 7:

(a) Show that $u(x,y) = -e^{-x}\sin(y) + x$ is harmonic on $\mathbb{R}^2$.

$u_x = e^{-x}\sin(y) + 1$ $\quad$ $u_y = -e^{-x}\cos(y)$

$u_{xx} = -e^{-x}\sin(y)$ $\quad$ $u_{yy} = e^{-x}\sin(y)$

$\therefore\ u_{xx} + u_{yy} = 0.$

[3]

(b) Find a harmonic conjugate $v(x,y)$ for $u(x,y)$ given in part (a)

$u_x = v_y \Rightarrow v_y = e^{-x}\sin(y) + 1 \Rightarrow v = -e^{-x}\cos(y) + y + g(x)$

$u_y = -v_x \Rightarrow -e^{-x}\cos(y) = -\left[e^{-x}\cos(y) + 0 + g'(x)\right]$

$\Rightarrow\ g'(x) = 0$

$\Rightarrow\ g(x) = C.$

$\therefore\ v(x,y) = -e^{-x}\cos(y) + y + C.$

[5]

(c) Express the analytic function $f(z) = u(x,y) + iv(x,y)$ as a function of z only.

$f(z) = [-e^{-x}\sin(y) + x] + i[-e^{-x}\cos(y) + y + C]$

$= -[e^{-x}\sin(y) + ie^{-x}\cos(y)] + x + iy + iC$

$= -i[e^{-x}\cos(y) - ie^{-x}\sin(y)] + x + iy + iC$

$= -i[e^{-x}\cos(-y) + ie^{-x}\sin(-y)] + x + iy + iC$

[2]

$= \boxed{-ie^{-z} + z + K}$, where $K = iC$.