# Math 372 - Introductory Complex Variables

G.Pugh

Feb 27 2017

# Recap of Last Day

**Theorem:** Let $f$ be continuous on the directed smooth curve $\gamma$ having admissible parametrization $z(t), \;\; a \leq t \leq b$. Then

$$\int_{\gamma} f(z)\, dz = \int_a^b f(z(t)) z'(t)\, dt$$

**Proof:** Let $\mathcal{P}_n = \{z(t_0), z(t_1), \ldots, z(t_n)\}$ be a partition of $\gamma$ and $\Delta z_k = z_k - z_{k-1}$, $\Delta t_k = t_k - t_{k-1}$.

Since $z(t)$ is differentiable,

$$z'(t_k) - \frac{\Delta z_k}{\Delta t_k} = \epsilon_k$$

where $\epsilon_k \to 0$ as $\Delta t_k \to 0$.

continued...

# Recap of Last Day, continued

So

$$\Delta z_k = z'(t_k)\Delta t_k - \epsilon_k \Delta t_k$$

So

$$\begin{aligned}\sum_{k=1}^{n} f(z_k)\Delta z_k &= \sum_{k=1}^{n} f(z(t_k))[z'(t_k)\Delta t_k - \epsilon_k \Delta t_k] \\ &= \sum_{k=1}^{n} f(z(t_k))z'(t_k)\Delta t_k - \sum_{k=1}^{n} f(z(t_k))\epsilon_k \Delta t_k\end{aligned}$$

continued...

# Recap of Last Day, continued

Now let $M_f = \max_{a \le t \le b} |f(z(t))|$ and $M_\epsilon = \max_{1 \le k \le n} |\epsilon_k|$. Then

$$
\begin{aligned}
\left| \sum_{k=1}^{n} f(z_k)\Delta z_k - \sum_{k=1}^{n} f(z(t_k))z'(t_k)\Delta t_k \right| &= \left| \sum_{k=1}^{n} f(z(t_k))\epsilon_k \Delta t_k \right| \\
&\le M_f M_\epsilon \sum_{k=1}^{n} \Delta t_k \\
&= M_f M_\epsilon (b-a)
\end{aligned}
$$

Letting $n \to \infty$ and $\mu(\mathcal{P}_n) \to 0$, $M_\epsilon \to 0$, leaving

$$
\left| \int_\gamma f(z)\,dz - \int_a^b f(z(t))z'(t)\,dt \right| = 0
$$

# Also from 4.2:

**Theorem:** Let $f$ be continuous on the contour $\Gamma$. Then

$$\left| \int_\gamma f(z)\, dz \right| \leq \left( \max_{z \in \Gamma} |f(z)| \right) \ell(\Gamma)$$

**Proof:** For any Riemann sum approximating the integral we have

$$\begin{aligned} \left| \sum_{k=1}^{n} f(c_k) \Delta z_k \right| &\leq \sum_{k=1}^{n} |f(c_k)| |\Delta z_k| \\ &\leq \max_{z \in \Gamma} |f(z)| \sum_{k=1}^{n} |\Delta z_k| \end{aligned}$$

Now let $n \to \infty$ and $\mu(\mathcal{P}_n) \to 0$.

# 4.3 - Independence of Path

# The Main Result

- Under certain conditions $\int_\Gamma f(z)\,dz$ can be computed without parametrizing $\Gamma$ .

- **Theorem:** Suppose that $f$ is continuous and has antiderivative $F$ throughout a domain $D$. Then for any contour $\Gamma$ with initial point $z_I$ and terminal point $z_T$,

$$\int_\Gamma f(z)\,dz = F(z_T) - F(z_I)$$

That is, the value of the integral is independent of the path $\Gamma$ joining the initial and terminal points.

# Corollary to the main result

**Corollary:** Suppose that $f$ is continuous and has antiderivative $F$ throughout the domain $D$. Then for any closed contour $\Gamma$ (so that $z_I = z_T$),

$$\int_\Gamma f(z)\,dz = F(z_T) - F(z_I) = 0$$

# Proof of Main Result

Suppose that $f$ is continuous and has antiderivative $F$ throughout a domain $D$, and let $\Gamma$ be a contour with initial point $z_I$ and terminal point $z_T$.

Suppose $\Gamma = \gamma_1 + \gamma_2 + \cdots + \gamma_n$ where each $\gamma_k$ is a smooth arc or smooth closed curve.

For each $j = 1, \ldots, n$, let $z_k(t)$ be an admissible parametrization of $\gamma_k$, where $a_k \leq t \leq b_k$. Notice: $z_k(b_k) = z_{k+1}(a_{k+1})$ .

continued…

# Proof of Main Result, continued

Now, calculate:

$$\begin{aligned}
\int_\Gamma f(z)\,dz &= \sum_{k=1}^{n} \int_{\gamma_k} f(z)\,dz \\
&= \sum_{k=1}^{n} \int_{a_k}^{b_k} f(z_k(t)) z_k'(t)\,dt \\
&= \sum_{k=1}^{n} [F(z_k(t)]_{a_k}^{b_k} \\
&= \sum_{k=1}^{n} [F(z_k(b_k)) - F(z_k(a_k))]
\end{aligned}$$

continued...

# Proof of Main Result, continued

This is a telescoping sum:

$$
\begin{aligned}
& \sum_{k=1}^{n} [F(z_k(b_k)) - F(z_k(a_k))] \\
= \quad & [F(z_1(b_1)) - F(z_1(a_1))] + [F(z_2(b_2)) - F(z_2(a_2))] + \cdots \\
& + [F(z_{n-1}(b_{n-1})) - F(z_{n-1}(a_{n-1}))] + [F(z_n(b_n)) - F(z_n(a_n))] \\
= \quad & F(z_n(b_n)) - F(z_1(a_1)) \\
= \quad & F(z_T) - F(z_I)
\end{aligned}
$$

# Existence of Antiderivatives

**Theorem:** If $f$ is continuous in a domain $D$ and contour integrals of $f$ are independent of path in $D$, then $f$ has an antiderivative in $D$.

**Proof:** Let $z_0 \in D$ be fixed and consider

$$F(z) = \int_{\Gamma_1} f(w)\, dw$$

where $\Gamma_1$ is any contour from $z_0$ to $z$.We must show that $F'(z) = f(z)$

continued...

# Existence of Antiderivatives, continued

Let $\Gamma_2$ be the line segment from $z$ to $z + \Delta z$. Then

$$
\begin{aligned}
F'(z) &= \lim_{\Delta z \to 0} \frac{F(z + \Delta z) - F(z)}{\Delta z} \\
&= \lim_{\Delta z \to 0} \frac{1}{\Delta z} \left[ \int_{\Gamma_1 + \Gamma_2} f(w)\, dw - \int_{\Gamma_1} f(w)\, dw \right] \\
&= \lim_{\Delta z \to 0} \frac{1}{\Delta z} \int_{\Gamma_2} f(w)\, dw
\end{aligned}
$$

continued...

# Existence of Antiderivatives, continued

On $\Gamma_2$, $w(t) = z + t\Delta z$, $w'(t) = \Delta z$, where $0 \leq t \leq 1$. So

$$
\begin{aligned}
F'(z) &= \lim_{\Delta z \to 0} \frac{1}{\Delta z} \int_0^1 f(z + t\Delta z)\Delta z \, dt \\
&= \lim_{\Delta z \to 0} \int_0^1 f(z + t\Delta z) \, dt \\
&= f(z)
\end{aligned}
$$

by the continuity of $f$.

# One last theorem

**Theorem:** If $f$ is continuous in a domain $D$ and $\int_{\Gamma} f(z)\,dz = 0$ for every loop (closed contour) $\Gamma$ in $D$ then contour integrals of $f$ in $D$ are independent of path.

**Proof:** Suppose $\Gamma_1$ and $\Gamma_2$ are two contours from point $z_I$ to point $z_T$. Then $-\Gamma_2$ is a contour from $z_T$ to $z_I$, so $\Gamma_1 + (-\Gamma_2)$ is a loop and

$$\int_{\Gamma_1+(-\Gamma_2)} f(z)\,dz = 0$$

continued...

# One last theorem, continued

Thus

$$\int_{\Gamma_1} f(z)\,dz + \int_{-\Gamma_2} f(z)\,dz = 0$$

so

$$\int_{\Gamma_1} f(z)\,dz = -\int_{-\Gamma_2} f(z)\,dz$$

so

$$\int_{\Gamma_1} f(z)\,dz = \int_{\Gamma_2} f(z)\,dz$$

# In Summary

We have shown:

**Theorem:** Suppose $f$ is continuous in a domain $D$. The following are equivalent (each statement implies the others):

(i) $f(z)$ has an antiderivative in $D$

(ii) $\int_{\Gamma} f(z)\,dz = 0$ for every loop in $D$

(iii) Contour integrals are independent of path in $D$