Math 372 - Complex Analysis

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Elementary Functions of Complex Analysis

Polynomials

- $p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$ where $a_k \in \mathbb{C}$
  Here $p(z)$ has ***degree*** n (assuming $a_n \neq 0$)

- **Theorem** (*Fundamental Theorem of Algebra*): Every nonconstant polynomial with complex coefficients has at least one zero in $\mathbb{C}$.

Polynomials, cont'd.

- Consequently, with the help of the division algorithm, any polynomial over $\mathbb{C}$ can be factored into linear factors

$$p(z) = a_n(z - z_1)(z - z_2) \cdots (z - z_n)$$

- Why? Suppose $p(z)$ has degree n and a zero $z = z_1$. Using the division algorithm,

$$p(z) = (z - z_1)q_1(z) + r(z)$$

  where the degree of the quotient factor $q_1(z)$ is $n - 1$, and that of $r(z)$ is strictly less than that of $z - z_1$. That is, $r(z) = k$ for some complex constant k. Now substitute $z = z_1$ to get

$$0 = p(z_1) = (z_1 - z_1)q_1(z_1) + k$$

  so that $k = 0$

Polynomials, cont'd.

- So
$$p(z) = (z - z_1)q_1(z)$$
Continue applying this argument to the quotient factor, reducing its degree at each step until it reaches a constant a_n:

$$\begin{aligned} p(z) &= (z - z_1)q_1(z) \\ &= (z - z_1)(z - z_2)q_2(z) \\ &= \cdots \\ &= (z - z_1)(z - z_2)\cdots(z - z_n)a_n \end{aligned}$$

Rational Functions

- Ratios of polynomials:

$$R_{m,n}(z) = \frac{p(z)}{q(z)} = \frac{a_0 + a_1 z + a_2 z^2 + \cdots + a_m z^m}{b_0 + b_1 z + b_2 z^2 + \cdots + b_n z^n}$$

- If $p(z)$ and $q(z)$ have no common factors, the zeros of $q(z)$ are called ***poles*** of $R_{m,n}(z)$

- Example: $f(z) = \dfrac{z^2 + 4}{(z-2)(z-3)^2}$ has a pole of order (or multiplicity) 1 at $z = 2$ and a pole of order 2 at $z = 3$.

Partial Fraction Decomposition

- The partial fraction decomposition result from calculus extends to rational functions over $\mathbb{C}$, but is simplified since all factors of denominator are linear.
- **Theorem** (*Partial Fraction Decomposition*): Suppose

$$R_{m,n}(z) = \frac{a_0 + a_1 z + a_2 z^2 + \cdots + a_m z^m}{b_n(z - z_1)^{d_1}(z - z_2)^{d_2} \cdots (z - z_r)^{d_r}}$$

where $d_1 + d_2 + \cdots + d_r = n > m$. Then

$$\begin{aligned} R_{m,n}(z) = & \frac{A_0^{(1)}}{(z - z_1)^{d_1}} + \frac{A_1^{(1)}}{(z - z_1)^{d_1 - 1}} + \cdots + \frac{A_{d_1 - 1}^{(1)}}{(z - z_1)} \\ & + \frac{A_0^{(2)}}{(z - z_2)^{d_2}} + \frac{A_1^{(2)}}{(z - z_2)^{d_2 - 1}} + \cdots + \frac{A_{d_2 - 1}^{(2)}}{(z - z_2)} \\ & + \cdots + \frac{A_0^{(r)}}{(z - z_r)^{d_r}} + \frac{A_1^{(r)}}{(z - z_r)^{d_r - 1}} + \cdots + \frac{A_{d_r - 1}^{(r)}}{(z - z_r)} \end{aligned}$$

Example: Partial Fraction Decomposition

Example: Determine the partial fraction decomposition of

$$f(z) = \frac{z^2 + 4}{(z-2)(z-3)^2}$$

Solution: Here $\dfrac{z^2+4}{(z-2)(z-3)^2} = \dfrac{a}{z-2} + \dfrac{b}{(z-3)^2} + \dfrac{c}{(z-3)}$

- $a = \lim_{z\to 2}(z-2)f(z) = \lim_{z\to 2} \dfrac{z^2+4}{(z-3)^2} = 8$
- $b = \lim_{z\to 3}(z-3)^2 f(z) = \lim_{z\to 3} \dfrac{z^2+4}{z-2} = 13$
- $c = \lim_{z\to 3} \dfrac{d}{dz}[(z-3)^2 f(z)] = \lim_{z\to 3} \dfrac{d}{dz}\left[\dfrac{z^2+4}{z-2}\right]$
  $= \lim_{z\to 3}\left[\dfrac{(z-2)(2z)-(z^2+4)(1)}{(z-2)^2}\right] = -7$
- Therefore $f(z) = \dfrac{8}{z-2} + \dfrac{13}{(z-3)^2} + \dfrac{-7}{z-3}$

Coefficients of Partial Fraction Decomposition

In general, if A_k is the coefficient of $\dfrac{1}{(z-\zeta)^{d-k}}$ in the partial fraction decomposition of the rational function $f(z)$ then

$$A_k = \lim_{z\to\zeta} \frac{1}{k!} \frac{d^k}{dz^k} \left[(z-\zeta)^d f(z) \right]$$

Exponential, Sine and Cosine Functions

- Recall: $e^z = e^{x+iy} = e^x[\cos(y) + i\sin(y)]$

- e^z is entire: $\dfrac{d}{dz}[e^z] = e^z$

- Since $e^{z+2\pi i} = e^z e^{2\pi i} = e^z$, say that e^z ($= \exp(z)$) is ***periodic*** with ***period*** $2\pi i$.

Definition of Sine and Cosine Functions

We define

$$\sin(z) = \frac{e^{iz} - e^{-iz}}{2i} \quad \text{and} \quad \cos(z) = \frac{e^{iz} + e^{-iz}}{2}$$

Observe: e^z entire $\implies \sin(z)$ and $\cos(z)$ entire.

- $\frac{d}{dz}[\sin(z)] = \cos(z)$
- $\frac{d}{dz}[\cos(z)] = -\sin(z)$
- Many standard real properties still apply: $\sin(z + 2\pi) = \sin(z)$, $\sin^2(z) + \cos^2(z) = 1$, etc.
- But not all! For example, it is not the case that $|\sin(z)| \leq 1$ for all z:

$$|\sin(-2i)| = \left| \frac{e^{i(-2i)} - e^{-i(-2i)}}{2i} \right| > \frac{e^2 - 1}{2} > 1$$

Definitions of other Trigonometric Functions

We define

$$\tan(z) = \frac{\sin(z)}{\cos(z)} \qquad \cot(z) = \frac{\cos(z)}{\sin(z)}$$

$$\sec(z) = \frac{1}{\cos(z)} \qquad \csc(z) = \frac{1}{\sin(z)}$$

These functions have the same derivatives as their real analogues:

- $\frac{d}{dz}[\tan(z)] = \sec^2(z)$
- $\frac{d}{dz}[\cot(z)] = -\csc^2(z)$
- $\frac{d}{dz}[\sec(z)] = \sec(z)\tan(z)$
- $\frac{d}{dz}[\csc(z)] = -\csc(z)\cot(z)$

Hyperbolic Functions

We define

$$\sinh(z) = \frac{e^z - e^{-z}}{2} \quad \text{and} \quad \cosh(z) = \frac{e^z + e^{-z}}{2}$$

- Observe: e^z entire $\implies$ $\sinh(z)$ and $\cosh(z)$ entire.
- $\frac{d}{dz}[\sinh(z)] = \cosh(z)$
- $\frac{d}{dz}[\cosh(z)] = \sinh(z)$
- $\sinh(iz) = i\sin(z)$
- $\cosh(iz) = \cos(z)$