Math 372 - Introductory Complex Variables

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Recap of Last Day

1.4 - The Complex Exponential

- The representation $z = re^{i\theta}$ is a consequence of the definition
$$e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!}$$
This series converges absolutely for every $z \in \mathbb{C}$; more on this later.
- Letting $z = i\theta$ in this definition we find:
$$e^{i\theta} = \left(\sum_{k=0}^{\infty} \frac{\theta^{2k}}{(2k)!} \right) + i \left(\sum_{k=0}^{\infty} \frac{\theta^{2k+1}}{(2k+1)!} \right) = \cos\theta + i \sin\theta \ ,$$
- Also can show that
$$e^z e^w = e^{z+w} \ ,$$
and so for $p, q \in \mathbb{R}$
$$e^p e^{iq} = e^{p+iq}$$

Euler's Equation

- The equation $e^{i\theta} = \cos\theta + i\sin\theta$ is called Euler's equation
- Letting $\theta = \pi$ we find

$$e^{i\pi} = \cos\pi + i\sin\pi$$

  from which

$$e^{i\pi} + 1 = 0$$

- Called "The most beautiful theorem in mathematics" by some

Euler's Equation continued

- Recall that for $\theta \in \mathbb{R}$:

$$\cos(-\theta) = \cos\theta, \quad \sin(-\theta) = -\sin\theta$$

- This gives

$$e^{i\theta} = \cos\theta + i\sin\theta$$
$$e^{-i\theta} = \cos\theta - i\sin\theta$$

- Now add and divide by 2 to find $\cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2}$.

- Subtract and divide by $2i$ to find $\sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i}$.

Periodicity of the complex exponential

- For $k \in \mathbb{Z}$,

$$e^{i(\theta+2k\pi)} = e^{i\theta} e^{i2k\pi} = e^{i\theta} \cdot 1 = e^{i\theta}$$

- Say that $e^{i\theta}$ is periodic with period 2π

Example

Find an identity which expresses $\sin(4\theta)$ in terms of $\sin\theta$ and $\cos\theta$.

1.5 - Powers and Roots

Powers

- For $n \in \mathbb{N}$ it is easy to define z^n:

$$z = |z|e^{i\theta}$$

so

$$z^n = |z|^n e^{in\theta}$$

- In fact, true for $n \in \mathbb{Z}$ if $z^{-n} = 1/z^n$.
- Value of z^n is the same regardless of branch of $\arg(z)$ used to define θ:

$$|z|^n e^{in(\theta+2k\pi)} = |z|^n e^{in\theta} e^{in2k\pi} = |z|^n e^{in\theta} \cdot 1$$

Roots

- For roots of complex numbers there is more to consider.
- **Definition:** For $m \in \mathbb{N}$, ζ is an m^{th} root of z if $\zeta^m = z$
- To find all m^{th} roots of a complex number $z = |z|e^{i\theta} \neq 0$, let $\zeta = \rho e^{i\phi}$ where $\rho > 0$.
- Then we must have $\rho^m e^{im\phi} = |z|e^{i\theta}$
- So $\rho = \sqrt[m]{|z|}$ and $e^{im\phi} = e^{i\theta}$

continued...

Roots, continued

- So $m\phi = \theta + 2k\pi$, where $k \in \mathbb{Z}$

- So $\phi = \dfrac{\theta}{m} + \dfrac{2k\pi}{m}$, where $k \in \mathbb{Z}$

- So all possible m^{th} roots of z are given by

$$\zeta = \sqrt[m]{|z|}e^{i(\theta+2k\pi)/m}, \quad k \in \mathbb{Z}$$

continued...

Roots, continued

- Notice: for $k = 0, 1, \ldots, m-1$ we have $0 \leq \frac{2k\pi}{m} < 2\pi$

- So
$$\zeta = \sqrt[m]{|z|}e^{i(\theta+2k\pi)/m}, \quad k = 0, 1, \ldots, m-1$$
represents m distinct m^{th} roots of z.

- Are these m roots the only ones? That is, what if $k \leq -1$ or $k \geq m$?

continued...

Roots, continued

- By the Division Algorithm there are integers q and r such that $k = qm + r$ where $0 \leq r \leq m - 1$
- So

$$\begin{aligned}
\zeta &= \sqrt[m]{|z|}e^{i(\theta+2k\pi)/m} \\
&= \sqrt[m]{|z|}e^{i(\theta+2(qm+r)\pi)/m} \\
&= \sqrt[m]{|z|}e^{i(\theta+2r\pi)/m}e^{i2qm\pi/m} \\
&= \sqrt[m]{|z|}e^{i(\theta+2r\pi)/m}
\end{aligned}$$

which, since $0 \leq r \leq m - 1$, is one of the roots we found already.

continued...

Roots, Conclusion

- **Theorem:** Let $m \geq 1$ be an integer and $z = re^{i\theta}$ with r, $\theta \in \mathbb{R}$, and where θ is given by any branch of $\arg(z)$. The m^{th} roots of z are given by

$$z^{1/m} = \sqrt[m]{|z|}e^{i(\theta+2k\pi)/m}, \quad k = 0, 1, \ldots, m-1$$

- **Corollary:** If m and n are positive integers with no common factors, then $(z^{1/n})^m = (z^m)^{1/n}$ and this common number, denoted by $z^{m/n}$ is given by

$$z^{m/n} = \sqrt[n]{|z|^m}e^{im(\theta+2k\pi)/n}, \quad k = 0, 1, \ldots, n-1$$

Example

Find all 6th roots of $z = \dfrac{2i}{1+i}$.

m^{th} roots of unity

- Consider the special case $z = 1 = e^{i\cdot 0}$
- Here the m^{th} roots are $\zeta = e^{i2k\pi/m}, \quad k = 0, 1, \ldots, m-1$
- This gives roots

$$
\begin{aligned}
(e^{i2\pi/m})^0 &= 1 \\
(e^{i2\pi/m})^1 &= \omega_m \\
(e^{i2\pi/m})^2 &= \omega_m^2 \\
&\vdots \\
(e^{i2\pi/m})^{m-1} &= \omega_m^{m-1}
\end{aligned}
$$

continued...

m^{th} roots of unity, continued

- So the m^{th} roots of unity are $1, \omega_m, \omega_m^2, \ldots, \omega_m^{m-1}$
- Here $\omega_m = e^{i2\pi/m}$ is called a primitive m^{th} root of unity since all the other m^{th} roots of 1 can be found by raising ω_m to positive integer powers.
- **Definition:** ω is called a primitive m^{th} root of unity if $\omega^m = 1$, but $\omega^q \neq 1$ for $1 \leq q \leq m - 1$.