Math 372 - Introductory Complex Variables

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4.1 - Contours

Smooth Arcs and Curves

- **Defintion:** The set of points $\gamma = \{z(t) \mid a \leq t \leq b\}$ is called a smooth arc if on $[a, b]$

  (i) $z(t)$ is continuously differentiable

  (ii) $z'(t) \neq 0$

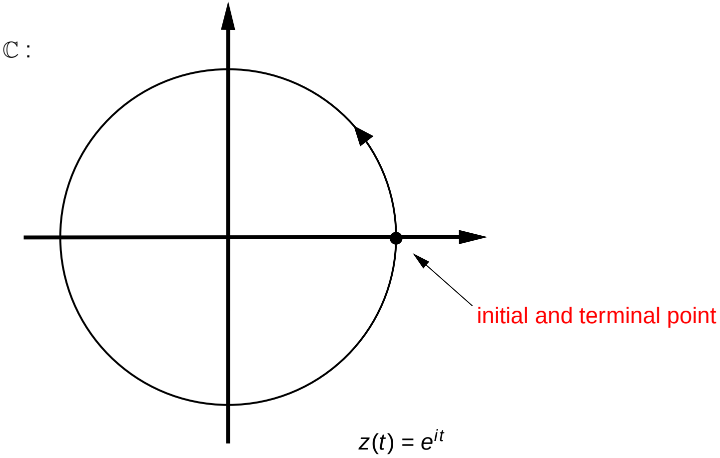
  (iii) $z(t)$ is one to one (i.e. if $z(t_1) = z(t_2)$ then $t_1 = t_2$.)

- γ is called a smooth closed curve if (i) and (ii) hold, and $z(t)$ is one to one on $[a, b)$ with $z(a) = z(b)$ and $z'(a) = z'(b)$

- Here $z(t)$ is called an admissible parametrization of γ .

Smooth Arcs and Curves: Examples

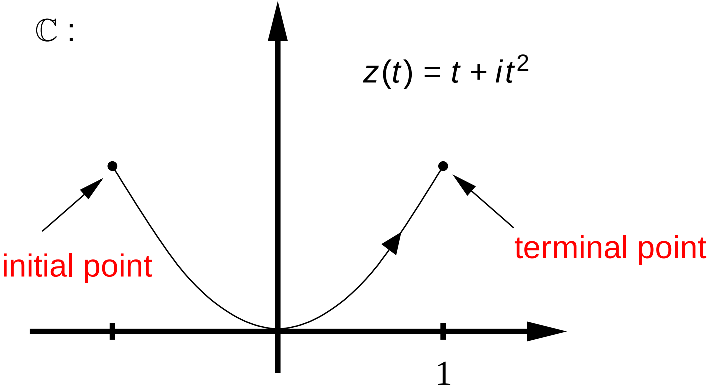
Example: $z(t) = e^{it}$, $0 \leq t \leq 2\pi$:



smooth closed curve

Smooth Arcs and Curves: Examples

Example: $z(t) = t + it^2$, $-1 \leq t \leq 1$:



smooth arc

Directed Smooth Arcs and Curves

Notice:

- The direction of increasing t is indicated on graphs, giving directed smooth curves

- There are many possible admissible parametrizations for a given arc or curve. For example,

$$z(t) = \cos(t) + i\cos^2 t, \quad -\pi \leq t \leq \pi$$

is an admissible parametrization for the second example above.

Contours

- **Defintion:** A contour Γ is either a single point, or a finite sequence of directed smooth curves $\gamma_1, \gamma_2, \ldots, \gamma_n$ such that the terminal point of γ_k coincides with the initial point of γ_{k+1} for each $k = 1, 2, \ldots, n-1$.

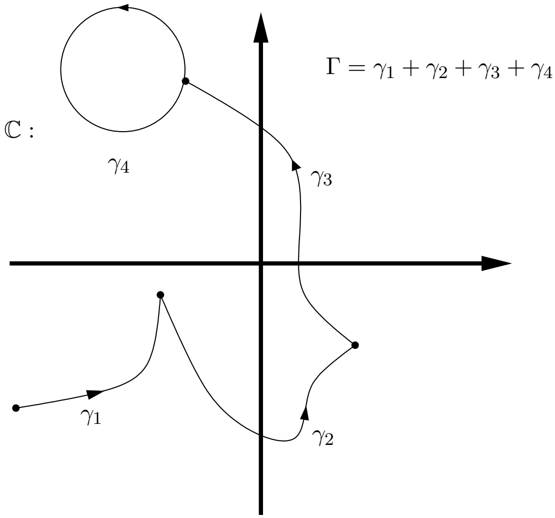
- To express that Γ consists of its set of directed smooth curves we write

$$\Gamma = \gamma_1 + \gamma_2 + \cdots + \gamma_n$$

- To express a contour consisting of the same points as Γ but traversed in the opposite direction we write $-\Gamma$.

Contours: Examples

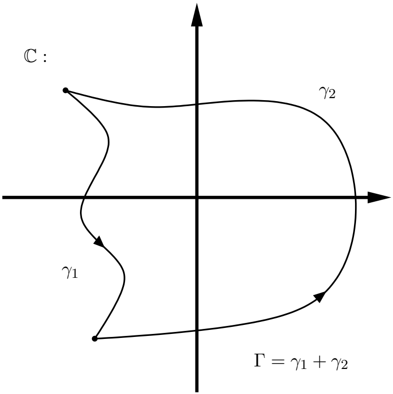
Example: a general contour:



A contour

Contours: Examples

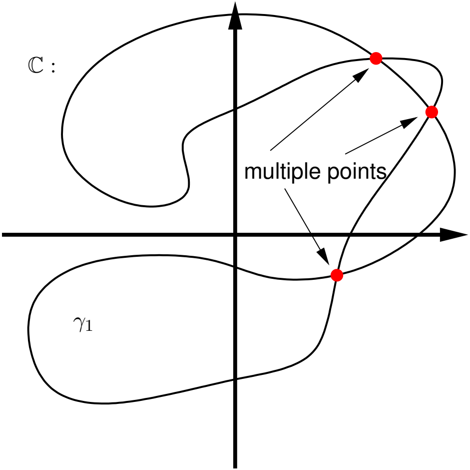
Definition: a contour Γ is called simple if for any admissible parametrization $z(t), a \le t \le b$, $z(t)$ is one to one on $[a, b]$. That is, if $z(t_1) = z(t_2)$ then either $t_1 = a$ and $t_2 = b$, or $t_1 = b$ and $t_2 = a$.



A simple closed contour

Contours: Examples

A contour that has multiple points is not simple:



A closed contour, but not simple

Jordan Curve Theorem

- **Theorem:** Any simple closed contour separates the plane into two domains, each having the curve as its boundary. The interior domain is bounded, while the exterior is unbounded.

- **Definition:** A directed simple closed contour is positively oriented if it is traversed in a counter-clockwise direction. That is, an observer traversing the contour in the positive direction will always have the interior to his or her left.

Length of a curve

If $z(t) = x(t) + iy(t),\ \ a \leq t \leq b$ is an admissible parametrization of the smooth curve γ, then from real variable theory $(x(t), y(t)),\ \ a \leq t \leq b$ is a parametrization of the corresponding curve in $\mathbb{R}^2$. The length of the curve is then

$$\ell(\gamma) = \int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt = \int_a^b \left| \frac{dz}{dt} \right| dt$$

An important parametrization

- The circle of radius ρ and centre z_0 (with positive orientation) has parametrization

$$z(t) = z_0 + \rho e^{it}, \quad 0 \leq t \leq 2\pi$$

- The length of this curve is

$$\begin{aligned}
\int_0^{2\pi} \left| \frac{dz}{dt} \right| \, dt &= \int_0^{2\pi} \left| \frac{d}{dt}[z_0 + \rho e^{it}] \right| \, dt \\
&= \int_0^{2\pi} \left| i\rho e^{it} \right| \, dt \\
&= \rho \int_0^{2\pi} \, dt \\
&= 2\pi\rho
\end{aligned}$$

4.2 - Contour Integrals

Formal Definition

- Let γ be a smooth curve and $f(z)$ a complex valued function defined on γ. We wish to define

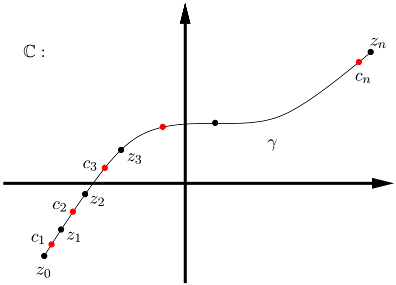
$$\int_\gamma f(z)\,dz$$

- Let $n \in \mathbb{N}$ and divide γ into n sub-arcs $\gamma_1, \gamma_2, \ldots, \gamma_n$ using points $z_0, z_1, \ldots, z_n$. Call this a partition $\mathcal{P}_n$ with mesh $\mu(\mathcal{P}_n) = \max_{1 \le k \le n} \ell(\gamma_k)$
- From each sub-arc γ_k select any point c_k and construct the Riemann Sum

$$S(\mathcal{P}_n) = \sum_{k=1}^{n} f(c_k)(z_k - z_{k-1})$$

continued...

Formal definition, continued



A partition of γ

- If $\lim\limits_{\substack{n\to\infty \\ \mu(\mathcal{P}_n)\to 0}} S(\mathcal{P}_n) = L$ and this limit is the same for every sequence of Riemann Sums then we define

$$\int_\gamma f(z)\,dz = L$$

and say that f is integrable along γ.

Computing Definite Integrals

- Notice: if γ is a real interval $[a, b]$ and f is real valued on γ, then
$$\int_\gamma f(z)\,dz = \int_a^b f(t)\,dt$$

- Many of the standard proofs for real variables carry over to contour integrals.

- **Theorem:** If f is continuous along the directed smooth curve γ then f is integrable along γ.

Computing Definite Integrals, continued

- **Theorem:** If γ is the real interval $[a, b]$ and $f(z) = u(z) + iv(z)$ where u and v are real and continuous then

$$\int_\gamma f(z)\,dz = \int_a^b f(t)\,dt = \int_a^b u(t)\,dt + i\int_a^b v(t)\,dt$$

Furthermore, if there is an antiderivative F such that $F'(t) = f(t)$ on $[a, b]$ then

$$\int_\gamma f(z)\,dz = \int_a^b f(t)\,dt = F(b) - F(a)$$

Computing Definite Integrals, continued

Theorem: Let f be continuous on the directed smooth curve γ having admissible parametrization $z(t), \;\; a \leq t \leq b$. Then

$$\int_\gamma f(z)\,dz = \int_a^b f(z(t))z'(t)\,dt$$