1. Let $m \in \mathbb{N}$ and $1 \leq j \leq m$. Define the projection function

$$p_j : \mathbb{R}^m \to \mathbb{R} \; : \; (x_1, \ldots, x_m) \mapsto x_j \; .$$

Show that p_j is continuous.

2. Let $n \in \mathbb{N}$, $m \in \mathbb{N}$ and for each $1 \leq j \leq m$,

$$f_j : \mathbb{R}^m \to \mathbb{R} \; .$$

Show that

$$f : \mathbb{R}^m \to \mathbb{R}^n \; : \; x \mapsto (f_1(x), \ldots, f_n(x))$$

is continuous if and only if each f_j is continuous.

Note: Exercises 1 and 2 establish a result that we have used implicitly a few times already.

3. For $x, y \in \mathbb{R}^n$, define the finite line segment

$$[x, y] = \{x + t(y - x) \mid 0 \leq t \leq 1\} \; .$$

Show that if $(0, 0, \ldots, 0) \in [x, y]$ with $x \neq (0, 0, \ldots, 0)$ and $y \neq (0, 0, \ldots, 0)$, then for z not on the (infinite) line through x and y we have

$$(0, 0, \ldots, 0) \notin [x, z] \cup [z, y]$$

Note: This result was assumed in Exercise 2 of Assignment 3 but is perhaps not obvious, so let's prove it!

4. Derive the homeomorphism

$$f : S^1 - \{(0, 1)\} \to \mathbb{R} \; : \; (x, y) \mapsto \frac{2x}{1 - y}$$

described in Example 5.7 of the text and determine the formula for $f^{-1} : \mathbb{R} \to S^1 - \{(0, 1)\}$.

Note: It is not necessary to prove that these are continuous bijections: simply derive the formulas.

5. Extend the result from the previous exercise: derive the stereographic projection homeomorphism described in Example 5.7

$$f : S^2 - \{(0, 0, 1)\} \to \mathbb{R}^2$$

6. Derive the homeomorphism

$$f : D \to Q \; : \; (x, y) \mapsto \frac{\sqrt{x^2 + y^2}}{\max(|x|, |y|)}(x, y)$$

described in Example 5.8 of the text. Again here, it is not necessary to prove that these are continuous bijections: simply derive the formulas.

7. Textbook Exercise 5.2

8. Textbook Exercise 5.7