Question 1:

(a) Use the trapezoid rule on 2 subintervals to approximate $\int_0^2 \frac{x^4+3x^2}{6}\,dx$. Simplify your final answer.

$\Delta x = 1$

$f(x) = \dfrac{x^4+3x^2}{6}$

$T_2 = \dfrac{\Delta x}{2}\left[f(0) + 2f(1) + f(2)\right]$

$= \dfrac{1}{2}\left[0 + 2\cdot\dfrac{4}{6} + \dfrac{28}{6}\right]$

$= \boxed{3}$

[5]

(b) Suppose that the midpoint rule is instead used to approximate the integral in part (a). Determine the number of subintervals n that are required to guarantee that the resulting approximation error is less than or equal to 1/27 .

$f'(x) = \dfrac{4x^3+6x}{6}$

$f''(x) = \dfrac{12x^2+6}{6} = 2x^2+1$

$|f''(x)| \leq 2(2^2)+1 = 9$ on $[0,2]$

$\therefore K = 9$

We want $|E_{M_2}| \leq \dfrac{1}{27}$

$\Rightarrow \dfrac{K(b-a)^3}{24n^2} \leq \dfrac{1}{27}$

$\Rightarrow \dfrac{9(2-0)^3}{24n^2} \leq \dfrac{1}{27}$

$\Rightarrow \dfrac{3}{n^2} \leq \dfrac{1}{27}$

$\Rightarrow 81 \leq n^2$

$\therefore \boxed{n \geq 9}$

[5]

Question 2:

(a) Evaluate the following improper integral making proper use of required limits:

$$I = \int_0^4 \frac{\ln(x)}{\sqrt{x}}\,dx$$

$$I = \lim_{a\to 0^+} \int_a^4 \frac{\ln(x)}{\sqrt{x}}\,dx$$

For $I_1 = \int \frac{\ln(x)}{\sqrt{x}}\,dx$,

let $u = \ln(x)$ $dv = x^{-\frac{1}{2}}dx$

$du = \frac{1}{x}dx$ $v = 2x^{\frac{1}{2}}$

$\therefore I_1 = \int u\,dv$

$= uv - \int v\,du$

$= 2x^{\frac{1}{2}}\ln(x) - \int 2x^{\frac{1}{2}}\cdot\frac{1}{x}\,dx$

$= 2x^{\frac{1}{2}}\ln(x) - 2\int x^{-\frac{1}{2}}\,dx$

$= 2x^{\frac{1}{2}}\ln(x) - 4x^{\frac{1}{2}} + C.$

$\therefore I = \lim_{a\to0^+}\left[2x^{\frac{1}{2}}\ln(x) - 4x^{\frac{1}{2}}\right]_a^4$

$= \lim_{a\to0^+}\left[2\sqrt{4}\ln(4) - 4\sqrt{4} - 2a^{\frac{1}{2}}\ln(a) + \underset{\to 0}{4a^{\frac{1}{2}}}\right]$

Now $\lim_{a\to0^+} 2a^{\frac{1}{2}}\ln(a)$

$= \lim_{a\to0^+}\frac{2\ln(a)}{a^{-1/2}}$

$\overset{H}{=} \lim_{a\to0^+}\frac{2/a}{-\frac{1}{2}a^{-3/2}}$

$= \lim_{a\to0^+} -4a^{\frac{1}{2}}$

$= 0$

$\therefore I = 8[\ln 2 - 1]$

[5]

(b) Evaluate the following improper integral making proper use of required limits:

$$\int_{-\infty}^{\infty}\frac{x^2}{9+x^6}\,dx = \underbrace{\int_{-\infty}^{0}\frac{x^2}{9+(x^3)^2}\,dx}_{I_1} + \underbrace{\int_0^{\infty}\frac{x^2}{9+(x^3)^2}\,dx}_{I_2}.$$

For $I = \int \frac{x^2}{9+(x^3)^2}\,dx$,

let $u = x^3$, $du = 3x^2$.

$\therefore I = \frac{1}{3}\int\frac{1}{3^2+u^2}\,du$

$= \frac{1}{9}\arctan\left(\frac{u}{3}\right) + C$

$= \frac{1}{9}\arctan\left(\frac{x^3}{3}\right) + C$

$\therefore I_1 = \lim_{a\to-\infty}\int_a^0 \frac{x^2}{9+x^6}\,dx$

$= \lim_{a\to-\infty}\left[\frac{1}{9}\arctan\left(\frac{x^3}{3}\right)\right]_a^0$

$= \lim_{a\to-\infty}\left[\frac{1}{9}\cancelto{0}{\arctan(0)} - \frac{1}{9}\arctan\left(\frac{a^3}{3}\right)\right]$

$= \left(\frac{-1}{9}\right)\left(-\frac{\pi}{2}\right) = \frac{\pi}{18}$

$I_2 = \lim_{b\to\infty}\left[\frac{1}{9}\arctan\left(\frac{x^3}{3}\right)\right]_0^b$

$= \lim_{b\to\infty}\left[\frac{1}{9}\arctan\left(\frac{b^3}{3}\right) - \frac{1}{9}\arctan(0)\right]$

$= \left(\frac{1}{9}\right)\left(\frac{\pi}{2}\right)$

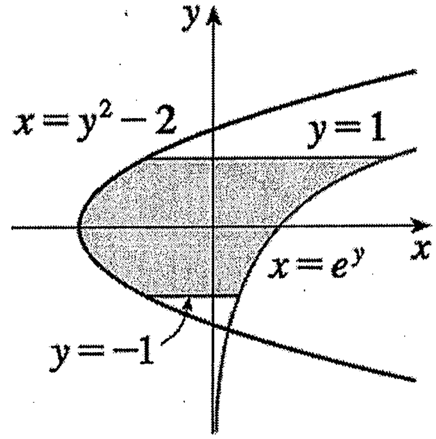
$= \frac{\pi}{18}$

$\therefore I = I_1 + I_2$

$= \boxed{\frac{\pi}{9}}$

[5]

Question 3: Determine the area of the shaded region:



$$A = \int_{y=-1}^{y=1} e^y - (y^2 - 2)\,dy$$

$$= \left[e^y - \frac{y^3}{3} + 2y \right]_{-1}^{1}$$

$$= \left(e - \frac{1}{3} + 2\right) - \left(e^{-1} + \frac{1}{3} - 2\right)$$

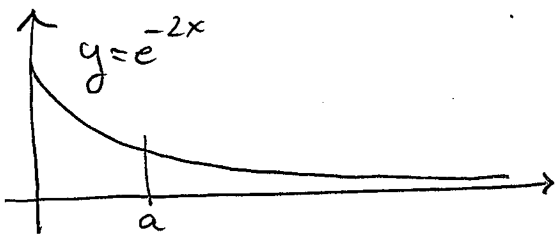
$$= e - \frac{1}{e} - \frac{2}{3} + 4$$

$$= \frac{3e^2 - 3 - 2e + 12e}{3e}$$

$$= \boxed{\frac{3e^2 + 10e - 3}{3e}}$$

[5]

Question 4: The region in the first quadrant between $y = e^{-2x}$ and the x-axis has finite area. Determine the value of a so that the vertical line $x = a$ divides this region into two smaller regions having equal areas.



Total area is $\int_0^\infty e^{-2x}\,dx$

$$= \lim_{b\to\infty} \int_0^b e^{-2x}\,dx$$

$$= \lim_{b\to\infty} \left[\frac{e^{-2x}}{-2} \right]_0^b$$

$$= \lim_{b\to\infty} \left[\underbrace{\frac{e^{-2b}}{-2}}_{\to 0} - \left(\frac{1}{-2}\right) \right] = \frac{1}{2}$$

Need $\int_0^a e^{-2x}\,dx = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)$

$$\left[\frac{e^{-2x}}{-2} \right]_0^a = \frac{1}{4}$$

$$\frac{e^{-2a}}{-2} + \frac{1}{2} = \frac{1}{4}$$

$$e^{-2a} = \frac{1}{2}$$

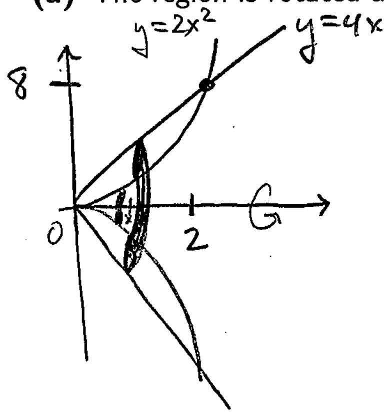
$$-2a = \ln\left(\tfrac{1}{2}\right)$$

$$a = -\frac{\ln(1/2)}{2} = \boxed{\frac{\ln(2)}{2}}$$

[5]

Question 5: This question deals with the region in the first quadrant that is bounded between $y = 2x^2$ and $y = 4x$. In each case the region is rotated about a central axis of rotation and you are asked to set up the integral representing the volume of the resulting solid. In each case DO NOT EVALUATE THE INTEGRAL, simply set it up.

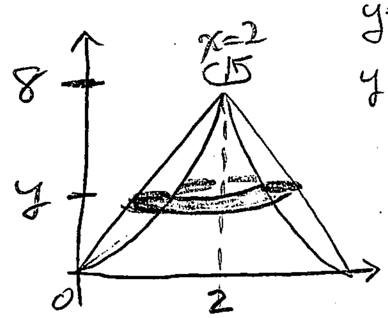
(a) The region is rotated about the x-axis. Set up an integral representing the volume of the resulting solid.



$$V = \int_0^2 \pi \left[(4x)^2 - (2x^2)^2 \right] dx$$

[3]

(b) The region is rotated about the vertical line $x = 2$. Set up an integral representing the volume of the resulting solid.



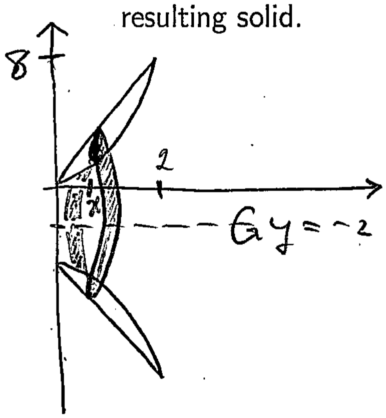
$y = 4x \Leftrightarrow x = \frac{y}{4}$

$y = 2x^2 \Leftrightarrow x = \sqrt{\frac{y}{2}}$

$$V = \int_0^8 \pi \left[\left(2 - \frac{y}{4}\right)^2 - \left(2 - \sqrt{\frac{y}{2}}\right)^2 \right] dy$$

[4]

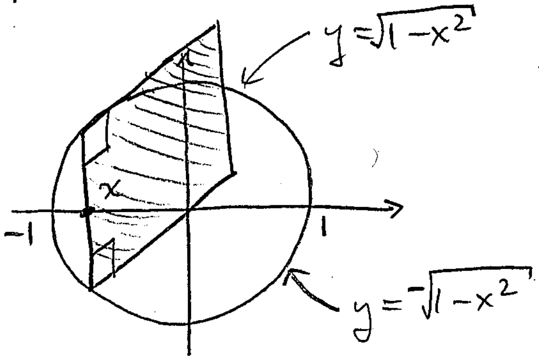
(c) The region is rotated about the horizontal line $y = -2$. Set up an integral representing the volume of the resulting solid.



$$V = \int_0^2 \pi \left[(4x+2)^2 - (2x^2+2)^2 \right] dx$$

[3]

Question 6: The flat base of a solid is a circle of radius 1. Parallel cross-sections perpendicular to the base are squares. Determine the volume of the solid.



$$A(x) = \left(2\sqrt{1-x^2}\right)^2 = 4(1-x^2)$$

$$\therefore V = \int_{-1}^{1} 4(1-x^2)\,dx$$

$$= 2\int_0^1 4(1-x^2)\,dx$$

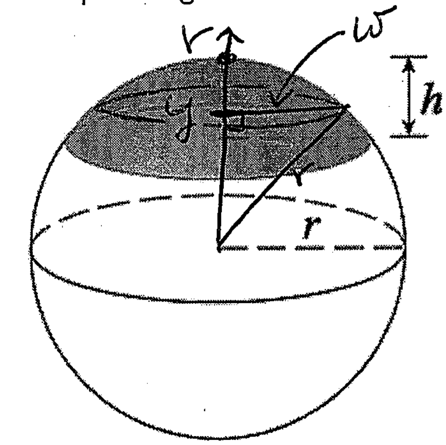
$$= 8\left[x - \frac{x^3}{3}\right]_0^1$$

$$= 8\left[1-\frac{1}{3}\right]$$

$$= \boxed{\frac{16}{3}}$$

[5]

Question 7: A sphere of radius r has a cap of height h as shown below. Determine the volume of the cap.



$$w^2 = r^2 - y^2$$

$$A(y) = \pi w^2$$

$$= \pi(r^2-y^2)$$

$$\therefore V = \int_{y=r-h}^{r} \pi(r^2-y^2)\,dy$$

$$= \pi\left[r^2y - \frac{y^3}{3}\right]_{r-h}^{r}$$

$$= \pi\left[\left(r^3 - \frac{r^3}{3}\right) - \left(r^2(r-h) - \frac{(r-h)^3}{3}\right)\right]$$

$$= \pi\left[\cancel{r^3} - \cancel{\frac{r^3}{3}} - \cancel{r^3} + \cancel{r^2h} + \cancel{\frac{r^3}{3}} - \cancel{\frac{3r^2h}{3}} + \frac{3rh^2}{3} - \frac{h^3}{3}\right]$$

$$= \boxed{\frac{\pi h^2(3r-h)}{3}}$$

[5]