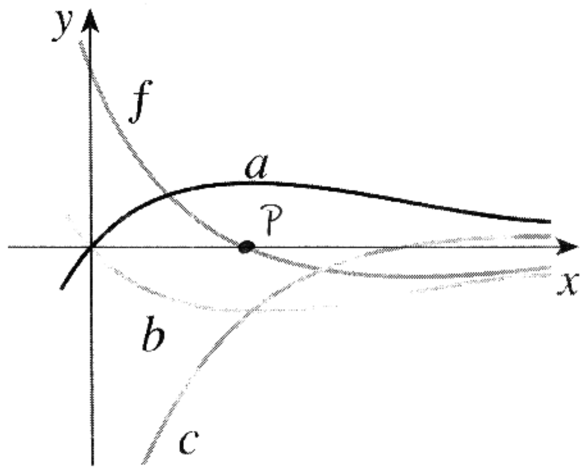
Question 1: Which of graphs a, b, or c below is that of the antiderivative of f? Briefly explain the reason for your choice.



Graph a

Only graph a has a positive tangent slope at $x=0$ and a zero tangent slope at $x=p$, so its derivative graph is consistent with that of f.

[5]

Question 2: Suppose an object moves along a straight line with acceleration at time t given by

$$a(t) = 4e^t - \frac{\sin(t)}{2}$$

where the initial velocity is $v(0) = 1/2$ m/s and the initial position is $s(0) = 0$ m. Find the position function $s(t)$.

$v(t) = 4e^t + \frac{1}{2}\cos(t) + C_1$

$v(0) = \frac{1}{2} \Rightarrow 4e^0 + \frac{1}{2}\cos(0) + C_1 = \frac{1}{2}$

$4 + \frac{1}{2} + C_1 = \frac{1}{2}$

$\therefore C_1 = -4$

$\therefore s(t) = 4e^t + \frac{1}{2}\sin(t) - 4t + C_2$

$s(0) = 0 \Rightarrow 4e^0 + \frac{1}{2}\cancel{\sin(0)}^{0} - \cancel{4 \cdot 0}^{0} + C_2 = 0$

$\therefore C_2 = -4$

$\therefore s(t) = 4e^t + \frac{1}{2}\sin(t) - 4t - 4$

[5]

Question 3: Use the definition of the definite integral in the form

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x$$

to evaluate

$$\int_0^4 \left(\frac{1}{4}x^3 - 2x + 1\right)\,dx$$

Carefully set up the Riemann sum and clearly show the steps of your simplification.

$\Delta x = \frac{4-0}{n} = \frac{4}{n}$

$x_i = 0 + i\left(\frac{4}{n}\right) = \frac{4i}{n}$

$f(x_i) = \frac{1}{4}(x_i)^3 - 2x_i + 1 = \frac{4^3 i^3}{4n^3} - 2\left(\frac{4i}{n}\right) + 1 = 16\frac{i^3}{n^3} - \frac{8i}{n} + 1$

$\therefore \int_0^4 \left(\frac{1}{4}x^3 - 2x + 1\right)dx$

$= \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\,\Delta x$

$= \lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{16i^3}{n^3} - \frac{8i}{n} + 1\right)\left(\frac{4}{n}\right)$

$= \lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{64i^3}{n^4} - \frac{32i}{n^2} + \frac{4}{n}\right)$

$= \lim_{n\to\infty}\left[\left(\frac{64}{n^4}\right)\left(\sum_{i=1}^{n} i^3\right) - \left(\frac{32}{n^2}\right)\left(\sum_{i=1}^{n} i\right) + \left(\frac{4}{n}\right)\left(\sum_{i=1}^{n} 1\right)\right]$

$= \lim_{n\to\infty}\left[\left(\frac{64}{n^4}\right)\frac{n^2(n+1)^2}{2^2} - \left(\frac{32}{n^2}\right)\frac{n(n+1)}{2} + \left(\frac{4}{n}\right)\cdot n\right]$

$= \lim_{n\to\infty}\left[\frac{64}{4}\cdot\frac{\cancel{n}}{n}\cdot\frac{\cancel{n}}{n}\cdot\underbrace{\frac{n+1}{n}}_{\to 1}\cdot\underbrace{\frac{n+1}{n}}_{\to 1} - \frac{32}{2}\cdot\frac{\cancel{n}}{n}\cdot\underbrace{\frac{n+1}{n}}_{\to 1} + 4\cdot\frac{\cancel{n}}{n}\right]$

$= 16 - 16 + 4$

$= \boxed{4}$

[10]

Question 4: Calculate the following definite integrals:

(a) $\displaystyle\int_1^9 \frac{3x-2}{\sqrt{x}}\,dx = \int_1^9 3x^{\frac{1}{2}} - 2x^{-\frac{1}{2}}\,dx$

$$= \cancel{3}\cdot\frac{2}{\cancel{3}}\left[x^{3/2}\right]_1^9 - 2\cdot\frac{2}{1}\left[x^{\frac{1}{2}}\right]_1^9$$

$$= 2\left[9^{3/2} - 1^{3/2}\right] - 4\left[9^{1/2} - 1^{1/2}\right]$$

$$= \boxed{44}$$

[3]

(b) $\displaystyle\int_0^1 \frac{4}{1+x^2}\,dx$

$$= 4\left[\arctan(x)\right]_0^1$$

$$= 4\left[\arctan(1) - \arctan(0)\right]$$

$$= 4\left[\frac{\pi}{4} - 0\right] = \boxed{\pi}$$

[2]

(c) $\displaystyle\int_0^3 (2x-3)(3x+2)\,dx$

$$= \int_0^3 6x^2 - 5x - 6\,dx$$

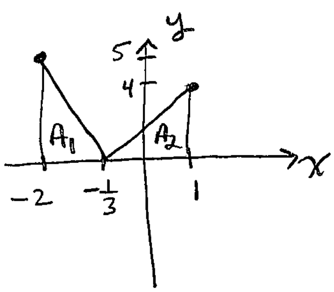
$$= \left[\frac{6x^3}{3} - \frac{5x^2}{2} - 6x\right]_0^3$$

$$= \frac{6(3^3)}{3} - \frac{5(3^2)}{2} - 6(3) - 0$$

$$= \boxed{\frac{27}{2}}$$

[2]

(d) $\displaystyle\int_{-2}^1 |3x+1|\,dx = A_1 + A_2$



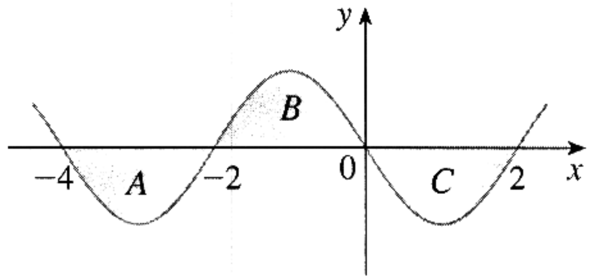
$$= \frac{1}{2}\left(-\frac{1}{3}+2\right)(5) + \frac{1}{2}\left(1+\frac{1}{3}\right)(4)$$

$$= \frac{25}{6} + \frac{16}{6}$$

$$= \boxed{\frac{41}{6}}$$

[3]

Question 5: In the graph of $y = g(x)$ below each of the regions A, B and C has area 3.



$$\therefore \int_{-4}^{2} g(x)\,dx = -3$$

$$\int_{-4}^{2} |g(x)|\,dx = 9$$

(a) Determine $\int_{-4}^{2} (2x + 5 - g(x))\,dx$ $= \int_{-4}^{2} 2x+5\,dx - \int_{-4}^{2} g(x)\,dx$

$$= \left[x^2+5x\right]_{-4}^{2} - (-3)$$

$$= (4+10) - (16-20) + 3 = \boxed{21}$$

[3]

(b) Determine $\int_{-4}^{2} |2g(x)|\,dx$ $= \int_{-4}^{2} |2|\,|g(x)|\,dx$

$$= 2\int_{-4}^{2} |g(x)|\,dx$$

$$= (2)(9) = \boxed{18}$$

[3]

(c) Determine g_{ave}, the average value of g over the interval $[-4, 2]$.

$$g_{ave} = \frac{1}{2-(-4)}\int_{-4}^{2} g(x)\,dx = \frac{1}{6}(-3) = \boxed{-\frac{1}{2}}$$

[1]

Question 6: Suppose $\int_{1}^{3} \frac{e^x}{x}\,dx = k$ where k is some constant. Determine the value of

$$\int_{3}^{1} \frac{e^x}{kx}\,dx$$

$$= -\int_{1}^{3} \frac{e^x}{kx}\,dx$$

$$= -\frac{1}{k}\int_{1}^{3} \frac{e^x}{x}\,dx$$

$$= \left(\frac{-1}{k}\right)(k)$$

$$= \boxed{-1}$$

[3]

Question 7: A fungal colony currently of mass 100 grams grows at a rate of $r(t) = 5 - 3\sqrt[3]{t}$ grams per week, where $t = 0$ weeks corresponds to the present.

(a) How large will the colony be after 8 weeks? Let $m(t)$ = mass at time t weeks.

$$m(8) - m(0) = \int_0^8 m'(t)\,dt$$

$$\therefore\ m(8) = m(0) + \int_0^8 5 - 3t^{1/3}\,dt$$

$$= 100 + 5[t]_0^8 - 3 \cdot \tfrac{3}{4}\left[t^{4/3}\right]_0^8$$

$$= 100 + (5)(8) - \tfrac{9}{4}(8^{4/3})$$

$$= 100 + 40 - 36$$

$$= \boxed{104\ g}$$

[3]

(b) Notice that as t increases from zero, the growth rate $r(t)$ is initially positive but then later becomes negative, meaning that the fungal colony initially grows in mass but later begins to decrease. Find the time require for the colony to return to its original mass of 100 grams.

Let T be the time required.

Then $m(T) - m(0) = 0$

$$\Rightarrow \int_0^T 5 - 3t^{1/3}\,dt = 0$$

$$\Rightarrow \left[5t - 3 \cdot \tfrac{3}{4} t^{4/3}\right]_0^T = 0$$

$$\Rightarrow 5T - \tfrac{9}{4} T^{4/3} = 0$$

$$\Rightarrow T\left(5 - \tfrac{9}{4}T^{1/3}\right) = 0$$

$$\Rightarrow 5 - \tfrac{9}{4}T^{1/3} = 0 \quad \text{since } T > 0$$

$$\Rightarrow T^{1/3} = (5)\left(\tfrac{4}{9}\right)$$

$$\Rightarrow \boxed{T = \left(\tfrac{20}{9}\right)^3 \text{ weeks}}$$

[3]

Question 8: Find a formula for $f(x)$ if f satisfies the following integral equation:

$$\int_1^x f(t)\,dt = \pi\left(x - \int_2^x t^2 f(t)\,dt\right)$$

$$\frac{d}{dx}\left(\int_1^x f(t)\,dt\right) = \frac{d}{dx}\left[\pi\left(x - \int_2^x t^2 f(t)\right)\right]$$

$$f(x) = \pi\left(1 - x^2 f(x)\right)$$

$$f(x) = \pi - \pi x^2 f(x)$$

$$f(x)(1 + \pi x^2) = \pi$$

$$\boxed{f(x) = \frac{\pi}{1 + \pi x^2}}$$

[4]