Question 1:

(a) Find the linear approximation $T_1(x)$ about $a = 1$ for $f(x) = \sqrt{x+3}$.

$f(1) = \sqrt{1+3} = 2$

$f'(x) = \frac{1}{2}(x+3)^{-\frac{1}{2}}$; $f'(1) = \frac{1}{2\sqrt{4}} = \frac{1}{4}$

$T_1(x) = f(a) + f'(a)(x-a)$

$\boxed{T_1(x) = 2 + \frac{1}{4}(x-1).}$

[5]

(b) Suppose you use your result in part (a) to approximate $\sqrt{4.1}$. Give an error bound on the resulting approximation.
Note: There is no need to actually calculate the approximation to $\sqrt{4.1}$, you are only being asked to determine a bound on the associated error. Express your answer as a single simplified fraction.

$\sqrt{4.1} = f(1.1)$

$R_1(x) = \frac{f''(z)}{2}(x-a)^2$ where $a=1$, $x=1.1$, $1<z<1.1$

$f'(z) = \frac{1}{2}(z+3)^{-\frac{1}{2}} \Rightarrow f''(z) = -\frac{1}{4}(z+3)^{-3/2} = \frac{-1}{4(z+3)^{3/2}}$.

$\therefore |R_2(x)| = \left| \frac{-1}{4(z+3)^{3/2}} \cdot \frac{1}{2} \cdot (x-a)^2 \right|$

$\therefore |R_2(1.1) = \left| \frac{-1}{4(z+3)^{3/2}} \cdot \frac{1}{2} \cdot (1.1-1)^2 \right|$

$< \frac{1}{4(1+3)^{3/2}} \cdot \frac{1}{2} \cdot \frac{1}{10^2} = \boxed{\frac{1}{6400}}$

[5]

Question 2:

(a) Find $T_3(x)$, the Taylor polynomial of degree 3, for $f(x) = x\ln(x) - x$ about $a = 1$.

$$f(1) = 1 \cdot \ln(1) - 1 = -1$$

$$f'(x) = 1 \cdot \ln(x) + x\frac{1}{x} - 1 = \ln(x) \;;\; f'(1) = \ln(1) = 0$$

$$f''(x) = \frac{1}{x} \;;\; f''(1) = \frac{1}{1} = 1$$

$$f'''(x) = \frac{-1}{x^2} \;;\; f'''(1) = -1$$

$$\therefore T_3(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3$$

$$\boxed{T_3(x) = -1 + \frac{1}{2}(x-1)^2 - \frac{1}{6}(x-1)^3}$$

[5]

(b) Give an error bound if $T_3(x)$ in part (a) is used to approximate $f(1/2)$.
Note: Again with this question: there is no need to actually calculate the approximation to $f(1/2)$, you are only being asked to determine a bound on the associated error. Express your answer as a single simplified fraction.

$$R_3(x) = \frac{f^{(4)}(z)}{4!}(x-a)^4 \quad \text{where} \quad x = \frac{1}{2},\ a = 1 \ \& \ \frac{1}{2} < z < 1.$$

$$f'''(z) = \frac{-1}{z^2} \Rightarrow f^{(4)}(z) = \frac{2}{z^3}.$$

$$\therefore R_3\left(\tfrac{1}{2}\right) = \frac{2}{z^3} \cdot \frac{1}{4!} \cdot \left(\tfrac{1}{2} - 1\right)^4$$

$$\therefore \left|R_3\left(\tfrac{1}{2}\right)\right| = \left|\frac{2}{z^3} \cdot \frac{1}{4!} \cdot \left(\frac{-1}{2}\right)^4\right|$$

$$< \frac{\cancel{2}}{\cancel{\left(\frac{1}{2}\right)^3}} \cdot \frac{1}{4!} \cdot \cancel{\left(\frac{1}{2}\right)^4}$$

$$= \boxed{\frac{1}{24}}$$

[5]

Question 3:

(a) Find the first three nonzero terms of the Maclaurin series for $f(x) = e^{(x^2)} \cos(3x)$.

$$e^{x^2} \cdot \cos(3x) = \left[1 + (x^2) + \frac{(x^2)^2}{2} + \frac{(x^2)^3}{3!} + \cdots\right]\left[1 - \frac{(3x)^2}{2} + \frac{(3x)^4}{4!} - \frac{(3x)^6}{6!} + \cdots\right]$$

$$= 1 + \left(1 - \frac{9}{2}\right)x^2 + \left(\frac{3^4}{4!} - \frac{9}{2} + \frac{1}{2}\right)x^4 + \cdots$$

$$= 1 - \frac{7}{2}x^2 + \left(\frac{27 - 36 + 4}{8}\right)x^4 + \cdots$$

$$= \boxed{1 - \frac{7}{2}x^2 - \frac{5}{8}x^4 + \cdots}$$

[6]

(b) The degree 6 term of the Maclaurin series for $f(x) = e^{(x^2)} \cos(3x)$ is $\frac{67x^6}{240}$. Determine $f^{(6)}(0)$, the sixth derivative of f at 0. Simplify your final answer.

$$\frac{f^{(6)}(0)\, x^6}{6!} = \frac{67x^6}{240}$$

$$\therefore \; f^{(6)}(0) = \frac{6! \, 67}{240} = \frac{\cancel{6 \cdot 5 \cdot 4} \cdot 3 \cdot \cancel{2} \cdot 67}{\cancel{6 \cdot 5 \cdot 4 \cdot 2}} = \boxed{201}$$

[4]

Question 4: The Maclaurin series for $e^{\arctan(x)}$ is

$$e^{\arctan(x)} = 1 + x + \frac{x^2}{2} - \frac{x^3}{6} - \frac{7x^4}{24} + \cdots$$

Use this to find the first three nonzero terms of the Maclaurin series for $g(x) = \dfrac{e^{\arctan(x)}}{1+x^2}$. (There are several ways to do this, but one way is much easier than the others.)

$$g(x) = \frac{d}{dx}\left[e^{\arctan(x)}\right] = \frac{d}{dx}\left[1 + x + \frac{x^2}{2} - \frac{x^3}{6} - \dots\right]$$

$$= 1 + \frac{2x}{2} - \frac{3x^2}{6} - \dots$$

$$= \boxed{1 + x - \frac{x^2}{2} - \dots}$$

[4]

Question 5: Evaluate the following limit:

$$\lim_{x\to 0} \frac{x^5 \sin(x) - e^{(x^6)} + 1}{x^8}$$

$$= \lim_{x\to 0} \frac{x^5\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots\right] - \left[\cancel{1} + (x^6) + \frac{(x^6)^2}{2} + \dots\right] + \cancel{1}}{x^8}$$

$$= \lim_{x\to 0} \frac{\left(\cancel{x^6} - \frac{x^8}{3!} + \frac{x^{10}}{5!} - \dots\right) - \left(\cancel{x^6} + \frac{x^{12}}{2} + \dots\right)}{x^8}$$

$$= \lim_{x\to 0} \frac{\cancel{x^8}\left(\frac{-1}{3!} + \frac{x^2}{5!} - \dots\right) - \cancel{x^8}\left(\frac{x^4}{2} + \dots\right)}{\cancel{x^8}}$$

$$= \frac{-1}{3!} = \boxed{\frac{-1}{6}}$$

[6]

Question 6: Find the radius of convergence R and open interval of convergence $\mathcal{I}$ for the power series

$$f(x) = \sum_{k=0}^{\infty} \frac{(-1)^k k! x^{2k}}{3^k}$$

$$\left.\right\} \begin{array}{l} a = 0 \\ u_k(x) = \dfrac{(-1)^k k!\, x^{2k}}{3^k}. \end{array}$$

$$\lim_{k\to\infty} \left| \frac{u_{k+1}(x)}{u_k(x)} \right| < 1$$

$$\Rightarrow \lim_{k\to\infty} \left| \frac{(-1)^{k+1}(k+1)!\, x^{2k+2}}{3^{k+1}} \cdot \frac{3^k}{(-1)^k k!\, x^{2k}} \right| < 1$$

$$\Rightarrow \lim_{k\to\infty} \left| \frac{(-1)^{k+1}}{(-1)^k} \cdot \frac{3^k}{3^{k+1}} \cdot \frac{(k+1)!}{k!} \cdot \frac{x^{2k+2}}{x^{2k}} \right| < 1$$

$$\Rightarrow \lim_{k\to\infty} \frac{1}{3} \cdot (k+1) \cdot |x|^2 < 1$$

$$\Rightarrow \infty < 1 \text{ unless } x = 0.$$

$$\boxed{\therefore R = 0, \quad \mathcal{I} \text{ does not exist.}}$$

[5]

Question 7: Find the radius of convergence R and open interval of convergence $\mathcal{I}$ for the power series

$$f(x) = \sum_{k=0}^{\infty} \frac{(x+2)^k}{(k+1)^3 2^k}$$

$$\left.\right\} \begin{array}{l} a = -2 \\ u_k(x) = \dfrac{(x+2)^k}{(k+1)^3 2^k} \end{array}$$

$$\lim_{k\to\infty} \left| \frac{u_{k+1}(x)}{u_k(x)} \right| < 1$$

$$\Rightarrow \lim_{k\to\infty} \left| \frac{(x+2)^{k+1}}{(k+2)^3\, 2^{k+1}} \cdot \frac{(k+1)^3 2^k}{(x+2)^k} \right| < 1$$

$$\Rightarrow \lim_{k\to\infty} \left| \underbrace{\frac{(k+1)^3}{(k+2)^3}}_{\to 1} \cdot \underbrace{\frac{2^k}{2^{k+1}}}_{= \frac{1}{2}} \cdot \frac{(x+2)^{k+1}}{(x+2)^k} \right| < 1$$

$$\Rightarrow \frac{1}{2}|x+2| < 1$$

$$\therefore |x+2| < 2$$

$$\text{or } |x-(-2)| < 2.$$

$$\boxed{\therefore R = 2, \quad \mathcal{I} = (-4, 0)}$$

[5]