**Question 1:** (Substitution Method)

**(a)** Determine $2\int \frac{\cos(\sqrt{x})}{2\sqrt{x}}\,dx. = I$

Let $u = \sqrt{x}$, $du = \frac{1}{2\sqrt{x}}\,dx$

$\therefore\; I = 2\int \cos(u)\,du = 2\sin(u) + C = \boxed{2\sin(\sqrt{x}) + C}$

[2]

**(b)** Determine $\int \frac{x^2+2x}{\sqrt{x^3+3x^2+5}}\,dx. = I$

Let $u = x^3+3x^2+5$

$du = (3x^2+6x)\,dx$

$\therefore\; I = \frac{1}{3}\int u^{-1/2}\,du$

$= \frac{1}{3}\cdot 2u^{1/2} + C$

$= \boxed{\frac{2}{3}\sqrt{x^3+3x^2+5} + C}$

[2]

**(c)** Evaluate $\int_0^1 (1-x)^{20}x\,dx. = I$

Let $u = 1-x \Rightarrow x = 1-u$

$du = -dx$

$x = 0 \Rightarrow u = 1$

$x = 1 \Rightarrow u = 0$

$\therefore\; I = -\int_1^0 u^{20}(1-u)\,du$

$= \int_0^1 u^{20} - u^{21}\,du$

$= \left[\frac{u^{21}}{21} - \frac{u^{22}}{22}\right]_0^1$

$= \frac{1}{21} - \frac{1}{22} = \boxed{\frac{1}{(21)(22)}}$

[3]

**(d)** Evaluate $\int_0^{\pi/4} \frac{\sec^2 x}{\sqrt{1-\tan^2 x}}\,dx. = I$

$u = \tan x$

$du = \sec^2 x\,dx$

$x = 0 \Rightarrow u = 0$

$x = \frac{\pi}{4} \Rightarrow u = 1$

$\therefore\; I = \int_0^1 \frac{1}{\sqrt{1-u^2}}\,du$

$= \left[\sin^{-1}(u)\right]_0^1$

$= \sin^{-1}(1) - \sin^{-1}(0)$

$= \boxed{\frac{\pi}{2}}$

[3]

**Question 2:** (Integration by Parts)

**(a)** Evaluate $\int x \sec^2 x \, dx$. $= \int u\,dv = uv - \int v\,du$

$u = x \quad dv = \sec^2 x \, dx$

$du = dx \quad v = \tan x$

$= x\tan x - \int \tan x \, dx$

$= x\tan x + \underbrace{\int \frac{-\sin x}{\cos x} \, dx}$

let $w = \cos x$

$dw = -\sin x \, dx$

$= \boxed{x\tan x + \ln|\cos x| + C}$

[5]

**(b)** Determine $\int_1^e x(\ln x)^2 \, dx$. $= \int_1^e u\,dv = [uv]_1^e - \int_1^e v\,du$

$u = (\ln x)^2 \quad dv = x\,dx$

$du = \frac{2\ln(x)}{x} \quad v = \frac{x^2}{2}$

$= \left[(\ln x)^2 \frac{x^2}{2}\right]_1^e - \int_1^e \frac{x^{\cancel{2}}}{\cancel{2}} \cdot \frac{\cancel{2}\ln(x)}{\cancel{x}} \, dx$

$= \frac{e^2}{2} - \int_1^e \underbrace{x\ln(x)\,dx}$

$u = \ln(x) \quad dv = x\,dx$

$du = \frac{1}{x}dx \quad v = \frac{x^2}{2}$

$= \frac{e^2}{2} - \left[\ln(x)\frac{x^2}{2}\right]_1^e + \int_1^e \frac{x^2}{2}\frac{1}{x}\,dx$

$= \frac{e^2}{2} - \frac{e^2}{2} + \frac{1}{2}\left[\frac{x^2}{2}\right]_1^e$

$= \boxed{\frac{e^2-1}{4}}$

[5]

**Question 3:** (Trigonometric Integrals)

**(a)** Evaluate $\int \tan^2(x)\sec^6(x)\,dx$.

$$= \int \tan^2(x)\left[\sec^2(x)\right]^2 \sec^2(x)\,dx$$

$$= \int \tan^2(x)\left[1+\tan^2(x)\right]^2 \sec^2(x)\,dx \quad \left.\begin{array}{l} u = \tan(x) \\ du = \sec^2(x)\,dx \end{array}\right.$$

$$= \int u^2(1+u^2)^2\,du$$

$$= \int u^2 + 2u^4 + u^6\,du$$

$$= \frac{u^3}{3} + \frac{2u^5}{5} + \frac{u^7}{7} + C$$

$$= \boxed{\frac{\tan^3(x)}{3} + \frac{2\tan^5(x)}{5} + \frac{\tan^7(x)}{7} + C}$$

[5]

**(b)** Determine $\int_0^{\pi} \sin^2(\theta/4) - \cos^2(\theta/4)\,d\theta$.

$$= \int_0^{\pi} \frac{1-\cos\left(2\cdot\frac{\theta}{4}\right)}{2} - \frac{1+\cos\left(2\cdot\frac{\theta}{4}\right)}{2}\,d\theta$$

$$= \int_0^{\pi} -\cos\left(\frac{\theta}{2}\right)d\theta$$

$$= -2\left[\sin\left(\frac{\theta}{2}\right)\right]_0^{\pi}$$

$$= -2\left[\cancel{\sin\left(\frac{\pi}{2}\right)}^{1} - \cancel{\sin(0)}^{0}\right]$$

$$= \boxed{-2}$$

[5]

**Question 4:** (Trigonometric Substitution) Determine

$$I = \int \frac{\sqrt{x^2-4}}{x^2}\,dx \qquad \text{Let } x = 2\sec\theta$$

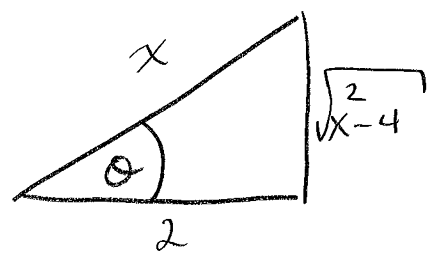
$$dx = 2\sec\theta\tan\theta\,d\theta$$

$$\therefore I = \int \frac{\sqrt{4\sec^2\theta - 4}}{4\sec^2\theta}\,2\sec\theta\tan\theta\,d\theta$$

$$= \int \frac{\cancel{2}\tan\theta}{\cancel{4}\sec^{\cancel{2}}\theta}\,\cancel{2}\,\cancel{\sec\theta}\tan\theta\,d\theta$$

$$= \int \frac{\tan^2\theta}{\sec\theta}\,d\theta$$

$$= \int \frac{\sec^2\theta - 1}{\sec\theta}\,d\theta$$




$$= \int \sec\theta - \cos\theta\,d\theta$$

$$= \ln\left|\sec\theta + \tan\theta\right| - \sin\theta + C$$

$$= \boxed{\ln\left|\frac{x}{2} + \frac{\sqrt{x^2-4}}{2}\right| - \frac{\sqrt{x^2-4}}{x} + C}$$

[10]

**Question 5:** (Partial Fractions) Determine

$$I = \int \frac{8(x-3)}{(x-1)(x+3)^2}\,dx$$

$$\frac{8x-24}{(x-1)(x+3)^2} = \frac{A}{x-1} + \frac{B}{x+3} + \frac{C}{(x+3)^2}$$

$$= \frac{A(x+3)^2 + B(x-1)(x+3) + C(x-1)}{(x-1)(x+3)^2}$$

$$= \frac{(A+B)x^2 + (6A+2B+C)x + (9A-3B-C)}{(x-1)(x+3)^2}$$

$$\left.\begin{array}{l} ① \quad A+B=0 \\ ② \quad 6A+2B+C=8 \\ ③ \quad 9A-3B-C=-24 \end{array}\right\}$$

① $\Rightarrow$ $B=-A$

② $\Rightarrow$ $C = 8-6A-2B = 8-6A-2(-A) = 8-4A$

③ $\Rightarrow$ $9A-3(-A)-(8-4A) = -24$

$$16A = -16$$

$$\therefore \quad A = -1$$

$$B = -(-1) = 1$$

$$C = 8-4(-1) = 12$$

$$\therefore I = \int \frac{-1}{x-1} + \frac{1}{x+3} + \frac{12}{(x+3)^2}\,dx$$

$$= \boxed{-\ln|x-1| + \ln|x+3| - \frac{12}{(x+3)} + C}$$

[10]