

# Math 371 - Introductory Real Analysis

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# Real Numbers

# Ordered Sets

- **Definition:**  $A$  is an **ordered set** if there exists a relation “ $<$ ” such that

- (i) For any  $x \in A$  and  $y \in A$  exactly one of

$$x < y, \quad x = y, \quad y < x$$

is true.

- (ii) If  $x < y$  and  $y < z$  then  $x < z$

- (iii)  $\leq, >, \geq$  have the standard meaning.

- **Examples:**  $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$  are ordered sets, but  $\mathbb{C}$  is not, nor is  $(\mathbb{Z}/n\mathbb{Z})$ .

# Bounded Sets: Definitions

Let  $E \subset A$  where  $A$  is an ordered set.

- ▶ **Definition:** If there is  $b \in A$  such that  $x \leq b$  for every  $x \in E$  we say that  $E$  is **bounded above** and  $b$  is **an upper bound** for  $E$ .
- ▶ **Definition:** If  $b_0$  is an upper bound for  $E$  and  $b_0 \leq b$  for every other upper bound  $b$ , then  $b_0$  is called **the least upper bound** of  $E$  or **the supremum** of  $E$ , and we write

$$b_0 = \sup E, \quad \text{read "sup of } E\text{"}$$

- ▶ **Definition:** If there is  $a \in A$  such that  $x \geq a$  for every  $x \in E$  we say that  $E$  is **bounded below** and  $a$  is **a lower bound** for  $E$ .
- ▶ **Definition:** If  $a_0$  is a lower bound for  $E$  and  $a_0 \geq a$  for every other lower bound  $a$ , then  $a_0$  is called **the greatest lower bound** of  $E$  or **the infimum** of  $E$ , and we write

$$a_0 = \inf E, \quad \text{read "inf of } E\text{"}$$

# Bounded Sets: Examples

- ▶ **Example:**  $E = \{2, 3, 4\} \subset \mathbb{N}$ .

1 is a lower bound for  $E$ , as is 2. 10 is an upper bound for  $E$ , as is 1000.

But  $\inf E = 2$ ,  $\sup E = 4$

- ▶ **Example:**  $E = \{\frac{1}{n} \mid n \in \mathbb{N}\} \subset \mathbb{Q}$ .

$\inf E = 0 \notin E$ ,  $\sup E = 1 \in E$

- ▶ **Example:**  $E = \{\sum_{k=0}^n \frac{1}{k!} \mid n \in \mathbb{N}\} \subset \mathbb{Q}$ .

$\inf E = 1 \in E$ ,  $\sup E$  does not exist in  $\mathbb{Q}$  ( $\sup E = e$  in fact).

# Least Upper Bound Property

- ▶ **Definition:** An ordered set  $A$  has the **least upper bound property** if every nonempty subset  $E \subset A$  that is bounded above has a least upper bound in  $A$ .

That is,  $\sup E$  exists and  $\sup E \in A$

- ▶ **Example:** We saw that  $\mathbb{Q}$  does not have the least upper bound property since  $\sup \left\{ \sum_{k=0}^n \frac{1}{k!} \mid n \in \mathbb{N} \right\} \notin \mathbb{Q}$ .
- ▶ To handle limits we need to extend  $\mathbb{Q}$  to a **field** which has the least upper bound property.

# Fields

**Definition:** A **field** is a set  $F$  together with two operations  $+$  and  $\cdot$  such that for any  $x, y, z \in F$ :

1.  $x + y \in F$
2.  $x + y = y + x$
3.  $(x + y) + z = x + (y + z)$
4. There exists a zero element  $0 \in F$  such that  $0 + x = x$
5. There exists an element  $-x$  such that  $x + (-x) = 0$
6.  $x \cdot y \in F$
7.  $x \cdot y = y \cdot x$
8.  $(x \cdot y) \cdot z = x \cdot (y \cdot z)$
9. There exists a unit element  $1 \in F$  such that  $1 \cdot x = x$
10. If  $x \neq 0$  there exists an element  $1/x$  such that  $(1/x) \cdot x = 1$
11.  $x \cdot (y + z) = x \cdot y + x \cdot z$
12.  $1 \neq 0$

# Examples of Fields

- ▶ Familiar:  $(\mathbb{Q}, +, \cdot)$  is a field
- ▶ More unusual: Recall that for  $a, p \in \mathbb{N}$ ,  
 $a \bmod p$  = remainder upon division of  $a$  by  $p$

Let  $p$  be a prime number and  $\mathbb{F} = \{1, 2, \dots, p-1\}$ .

For  $a, b \in \mathbb{F}$  define  $a +_{\mathbb{F}} b = a + b \bmod p$

define  $a \cdot_{\mathbb{F}} b = ab \bmod p$

Then  $(\mathbb{F}, +_{\mathbb{F}}, \cdot_{\mathbb{F}})$  is a field



# Ordered Fields

- ▶ **Definition:** An ordered set  $F$  is an **ordered field** if
  - ▶  $F$  is a field (satisfies the field axioms),
  - ▶  $x < y \implies x + z < y + z$
  - ▶  $x > 0$  and  $y > 0 \implies xy > 0$
- ▶  $(\mathbb{Q}, +, \cdot)$  is an ordered field, but  $(\mathbb{F}, +_{\mathbb{F}}, \cdot_{\mathbb{F}})$  is not.

# Ordered Fields

The usual notions of **positive** ( $x > 0$ ) and **negative** ( $x < 0$ ) are defined for ordered fields, and the familiar operations and results involving inequalities still hold:

**Proposition:** For  $x, y, z \in F$  an ordered field,

- ▶  $x > 0 \implies -x < 0$
- ▶  $x > 0$  and  $y < z \implies xy < xz$
- ▶  $x < 0$  and  $y < z \implies xy > xz$
- ▶  $x \neq 0 \implies x^2 > 0$
- ▶  $0 < x < y \implies 0 < 1/y < 1/x$

# The Real Numbers

- ▶ **Theorem:** There exists a unique ordered field  $\mathbb{R}$  with the least upper bound property such that  $\mathbb{Q} \subset \mathbb{R}$
- ▶ **Note:** There are several techniques for constructing  $\mathbb{R}$ . Two of the more popular are construction using Cauchy sequences, and construction using Dedekind cuts.
- ▶ In summary:

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$$

where  $\mathbb{Q}$  and  $\mathbb{R}$  are ordered fields, but only  $\mathbb{R}$  has the least upper bound property.

- ▶  $\mathbb{N}$ ,  $\mathbb{Z}$  and  $\mathbb{Q}$  are countably infinite, but  $\mathbb{R}$  is uncountable.
- ▶ The set of irrational numbers  $\mathbb{R} \setminus \mathbb{Q}$  is uncountable.