1. Suppose that $S \subset \mathbb{R}$ and that S is nonempty and bounded. Show that for every $\epsilon > 0$ there is some $x \in S$ such that
$$\sup(S) - \epsilon < x \leq \sup(S)$$
(We have used this result in proofs already without showing it explicitly. The proof is very short!)

2. Let $\{n_i\}_{i=1}^{\infty}$ be a subsequence of $\{n\}_{n=1}^{\infty}$. Show that $i \leq n_i$ for every $i \in \mathbb{N}$.
(A result that appears obvious, so let's prove it.)

The next two problems deal with intervals of $\mathbb{R}$. We didn't formally define intervals in class, but the meaning (as a set) is the same as that used in calculus. See the beginning of Section 1.4 of the text for the formal definition.

3. Show that
   (a) Every closed interval can be expressed as the intersection of a countable number of open intervals.
   (b) Every open interval can be expressed as the union of a countable number of closed intervals.

4. Show that if S is a set of disjoint open intervals in $\mathbb{R}$ then S is countable.

5. Find $\lim_{n\to\infty} 2^{-n}$ and prove your result. Do not make use of logarithms in your proof; use only techniques and results developed in the course so far.

6. Find $\lim_{n\to\infty} \frac{n}{n^2+1}$ and prove your result.

7. Suppose that $S \subset \mathbb{R}$ and that S is nonempty and bounded. Show that there exists a monotone increasing sequence $\{x_n\}_{n=1}^{\infty}$ such that $x_n \in S$ for each $n \in \mathbb{N}$ and $\lim_{n\to\infty} x_n = \sup(S)$.

8. Prove that there exists a sequence $\{x_n\}_{n=1}^{\infty}$ with the property that for each $y \in \mathbb{R}$ there is a subsequence $\{x_{n_i}\}_{i=1}^{\infty}$ with $x_{n_i} \to y$.

9. On Assignment 1 we proved that $n^2 < 2^n$ for $n \geq 5$. Prove that
$$\lim_{n\to\infty} \frac{n^2}{2^n} = 0 \ .$$