Question 1:

(a) Find (i) the radius of convergence and (ii) the open interval of convergence for the power series

$$\sum_{k=1}^{\infty} \frac{(-1)^k k(x-1)^k}{5^k}$$

Let $u_k(x) = \frac{(-1)^k k(x-1)^k}{5^k}$

$$\lim_{k\to\infty} \left| \frac{u_{k+1}(x)}{u_k(x)} \right| < 1 \Rightarrow \lim_{k\to\infty} \left| \frac{(-1)^{k+1}(k+1)(x-1)^{k+1}}{5^{k+1}} \cdot \frac{5^k}{(-1)^k k (x-1)^k} \right| < 1$$

$$\Rightarrow \lim_{k\to\infty} \underbrace{\left| \frac{(-1)^{k+1}}{(-1)^k} \right|}_{=1} \cdot \underbrace{\left| \frac{k+1}{k} \right|}_{\to 1} \cdot \underbrace{\left| \frac{5^k}{5^{k+1}} \right|}_{=\frac{1}{5}} \cdot \underbrace{\left| \frac{(x-1)^{k+1}}{(x-1)^k} \right|}_{=|x-1|} < 1$$

$$\Rightarrow \frac{|x-1|}{5} < 1$$

$$\Rightarrow |x-1| < 5$$

$$\boxed{\therefore R = 5, \quad I = (-4, 6).}$$

[5]

(b) Find (i) the radius of convergence and (ii) the open interval of convergence for the power series

$$\sum_{k=0}^{\infty} \frac{3^k x^k}{k!}$$

Let $u_k(x) = \frac{3^k x^k}{k!}$

$$\lim_{k\to\infty} \left| \frac{u_{k+1}(x)}{u_k(x)} \right| < 1 \Rightarrow \lim_{k\to\infty} \left| \frac{3^{k+1} x^{k+1}}{(k+1)!} \cdot \frac{k!}{3^k x^k} \right| < 1$$

$$\Rightarrow \lim_{k\to\infty} \underbrace{\left| \frac{3^{k+1}}{3^k} \right|}_{=3} \cdot \underbrace{\left| \frac{k!}{(k+1)!} \right|}_{=\frac{1}{k+1}} \cdot \underbrace{\left| \frac{x^{k+1}}{x^k} \right|}_{=x} < 1$$

$$\Rightarrow \lim_{k\to\infty} \frac{3x}{k+1} < 1$$

$$\Rightarrow 0 < 1 \Big\} \text{ True for every real number } x.$$

$$\boxed{\therefore R = \infty, \quad I = (-\infty, \infty).}$$

[5]

Question 2: Use the definition of the definite integral in the form

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x$$

to evaluate

$$\int_0^2 (2 - x + 3x^2)\,dx$$

Carefully set up the Riemann sum and clearly show the steps of your simplification.

Here $[a,b] = [0,2]$.

$\Delta x = \frac{b-a}{n} = \frac{2-0}{n} = \frac{2}{n}$

$x_i = a + i\Delta x = 0 + i\left(\frac{2}{n}\right) = \frac{2i}{n}$

$f(x_i) = 2 - x_i + 3(x_i)^2$

$= 2 - \frac{2i}{n} + 3\left(\frac{2i}{n}\right)^2$

$$\therefore \int_0^2 (2 - x + 3x^2)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x$$

$$= \lim_{n\to\infty}\sum_{i=1}^{n}\left[2 - \left(\frac{2i}{n}\right) + 3\left(\frac{2i}{n}\right)^2\right]\left(\frac{2}{n}\right)$$

$$= \lim_{n\to\infty}\sum_{i=1}^{n}\left[\frac{4}{n} - \frac{4}{n^2}i + \frac{24}{n^3}i^2\right]$$

$$= \lim_{n\to\infty}\left[\left(\frac{4}{n}\right)\sum_{i=1}^{n} 1 - \left(\frac{4}{n^2}\right)\left(\sum_{i=1}^{n} i\right) + \left(\frac{24}{n^3}\right)\left(\sum_{i=1}^{n} i^2\right)\right]$$

$$= \lim_{n\to\infty}\left[\left(\frac{4}{\cancel{n}}\right)(\cancel{n}) - \left(\frac{4}{n^2}\right)\left(\frac{n(n+1)}{2}\right) + \left(\frac{24}{n^3}\right)\left(\frac{n(n+1)(2n+1)}{6}\right)\right]$$

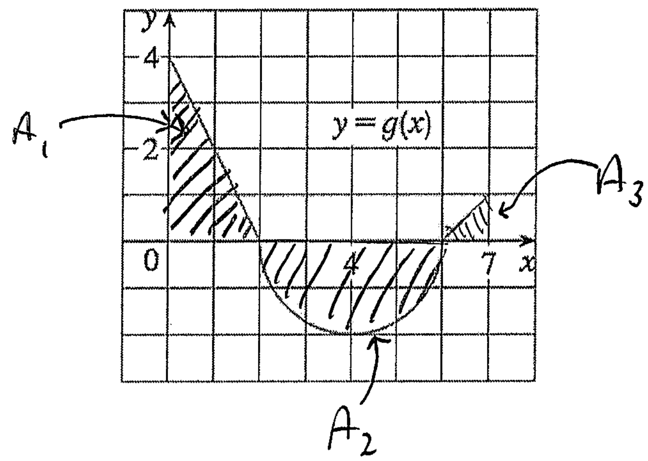
$$= \lim_{n\to\infty}\left[4 - \left(\frac{4}{2}\cdot\frac{\cancel{n}}{\cancel{n}}\cdot\frac{n+1}{n}\right) + \left(\frac{24}{6}\cdot\frac{n}{n}\cdot\frac{n+1}{n}\cdot\frac{2n+1}{n}\right)\right]$$

$= 4 - 2 + 8$

$= \boxed{10}$

[10]

Question 3: The graph of $y = g(x)$ over the interval $[0, 7]$ consists of two straight lines and a semicircle (half of a circle) as shown below. Determine $\displaystyle\int_0^7 g(x)\,dx$.



$$\int_0^7 g(x)dx = A_1 - A_2 + A_3$$

$$= \left(\tfrac{1}{2}\right)(2)(4) - \tfrac{1}{2}\pi\cdot(2^2) + \left(\tfrac{1}{2}\right)(1)(1)$$

$$= \frac{8 - 4\pi + 1}{2}$$

$$= \boxed{\frac{9-4\pi}{2}}$$

[5]

Question 4: The trunk of a growing tree has a circular cross-section with a radius that increases at a rate of $\dfrac{(5+t)}{1000}$ metres per year. If the trunk radius is currently 1/4 m, what will be the radius in 20 years?

$$r(20) - r(0) = \int_0^{20} r'(t)\,dt$$

$$\therefore\ r(20) = r(0) + \int_0^{20} \frac{5+t}{1000}\,dt$$

$$= \frac{1}{4} + \frac{1}{1000}\left[5t + \frac{t^2}{2}\right]_0^{20}$$

$$= \frac{1}{4} + \frac{100 + 200}{1000}$$

$$= \frac{1}{4} + \frac{3}{10}$$

$$= \boxed{\frac{11}{20}\,\text{m}}$$

[5]

Question 5:

(a) Find $\int (\sin(x) - 2e^x + \pi)\, dx$ $= \boxed{-\cos(x) - 2e^x + \pi x + C}$

[2]

(b) Find $\int_1^4 \frac{1+\sqrt{x}}{x}\, dx$

$$= \int_1^4 \frac{1}{x} + x^{-\frac{1}{2}}\, dx$$

$$= \left[\ln|x| + 2x^{1/2}\right]_1^4$$

$$= (\ln|4| + 2(4)^{1/2}) - (\cancel{\ln|1|}^{0} + 2(1)^{1/2})$$

$$= \boxed{\ln(4) + 2}$$

[2]

(c) Find $\int_{-1}^{1} x(1-x)^2\, dx$

$$= \int_{-1}^{1} x - 2x^2 + x^3\, dx$$

$$= \left[\frac{x^2}{2} - \frac{2x^3}{3} + \frac{x^4}{4}\right]_{-1}^{1}$$

$$= \left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right) - \left(\frac{1}{2} + \frac{2}{3} + \frac{1}{4}\right) = \boxed{-\frac{4}{3}}$$

[2]

Question 6: Find the value of c so that the average value of $f(x) = \sqrt{x}$ over $[0, 4]$ is equal to $f(c)$.

$$f_{ave} = \frac{1}{4}\int_0^4 x^{1/2}\, dx$$

$$= \frac{1}{4}\ \frac{2}{3}\left[x^{3/2}\right]_0^4$$

$$= \left(\frac{1}{4}\right)\left(\frac{2}{3}\right)(8)$$

$$= \frac{4}{3}$$

now solve $\sqrt{c} = \frac{4}{3}$

$\therefore \boxed{c = \frac{16}{9}}$

[4]

Question 7: Substitution Method:

(a) Find $\int x^2 e^{(x^3)}\,dx$ $\Big\}$ let $u = x^3$, $du = 3x^2\,dx$

$= \frac{1}{3}\int e^u\,du$

$= \frac{1}{3}e^u + C$

$= \boxed{\frac{1}{3}e^{x^3} + C}$

[3]

(b) Find $\int \frac{1}{x\ln(x)}\,dx$ $\Big\}$ let $u = \ln(x)$, $du = \frac{1}{x}\,dx$

$= \int \frac{1}{u}\,du$

$= \ln|u| + C$

$= \boxed{\ln|\ln(x)| + C}$

[3]

(c) Find $\int_1^e \frac{\cos(\pi\ln(x))}{x}\,dx$ $\Big\}$ let $u = \pi\ln(x)$, $du = \frac{\pi}{x}dx$ $\quad x=1 \Rightarrow u=0$, $x=e \Rightarrow u=\pi$

$= \frac{1}{\pi}\int_0^{\pi}\cos(u)\,du$

$= \frac{1}{\pi}\left[\sin(u)\right]_0^{\pi}$

$= \frac{1}{\pi}\left[\underbrace{\cancel{\sin(\pi)}}_{0} - \underbrace{\cancel{\sin(0)}}_{0}\right] = \boxed{0}$

[4]