1 Introduction

These notes give a brief introduction to the representation of functions using series (that is, the sum of an infinite number of terms.) For example, in a precalculus course you may have seen the ***geometric series***

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots$$

which is true for any value of x between -1 and 1. By writing equality here we mean that the difference between $1/(1-x)$ and $1+x+x^2+x^3+x^4+\cdots+x^n$ approaches zero as n grows to infinity. Many other functions have such representations. For example, the familiar exponential function has representation

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{2\cdot 3} + \frac{x^4}{2\cdot 3\cdot 4} + \cdots$$

while for the sine function,

$$\sin x = x - \frac{x^3}{2\cdot 3} + \frac{x^5}{2\cdot 3\cdot 4\cdot 5} - \cdots$$

These last two series are valid for all real numbers x.

These infinite series representations are important for a number of reasons. One is that they form the basis of the methods used to evaluate functions. At the most fundamental level, numerical calculation (by human or machine) involves only the operations of addition, subtraction, multiplication and division, the same four operations required to add up the terms in a series. If, for example, $\sin(37^\circ)$ is required in a calculation, how do we (or our calculators) come up with the decimal approximation 0.60181502315? The procedure used is based on the series representation of $\sin x$ above.

Another important reason for the study of series is found in the solution of real-world physics and engineering problems. Often the solutions to such problems cannot be expressed as finite combinations of elementary functions (polynomial, trigonometric, exponential, etc) and an infinite series is the only known solution representation.

2 Some Definitions

To describe our results we need a few definitions. These are likely already familiar to you but we include them here for review and reference.

Definition 1

An ***interval*** is a set of numbers which form a segment of the real number line. For example, $(0,1)$, $[0,1]$, $(-3,\infty)$, $(-\infty,\pi]$, $(-\infty,\infty)$ are all intervals.

An ***open interval*** is an interval which does not include its upper and lower boundaries and so uses only "(" and ")" in its representation. Examples of open intervals are

$$(0,1), (-2,\infty) \text{ and } (-\infty,\infty)\ .$$

A ***closed interval*** is an interval which includes it's upper and lower boundaries, such as $[0,1]$ or $[e,\pi]$.

The letter $\mathcal{I}$ is often used to denote a general interval.

Definition 2

A function f is ***differentiable*** at a point $x = a$ if $f'(a)$ exists, meaning that the limit defining the derivative

$$\lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

exists.

A function is **n-*times differentiable*** at a point $x = a$ if the n^{th} derivative $f^{(n)}(a)$ exists (and consequently $f'(a)$, $f''(a)$, ..., $f^{(n)}(a)$ all exist.)

A function is ***differentiable on an open interval*** $\mathcal{I}$ if it is differentiable at every point x in $\mathcal{I}$, and similarly for **n-*times differentiable on an open interval***.

For example, $f(x) = x^{7/3}$ is twice differentiable on $(-1, 1)$ since $f''(x) = (28/9)x^{1/3}$ exists for every x in $(-1, 1)$. However, $f(x) = x^{7/3}$ is not 3-times differentiable on $(-1, 1)$ since

$$\begin{aligned} f'''(0) &= \lim_{h\to 0} \frac{f''(0+h) - f''(0)}{h} \\ &= \lim_{h\to 0} \frac{(28/9)h^{1/3} - 0}{h} \\ &= \lim_{h\to 0} (28/9)h^{-2/3} \end{aligned}$$

fails to exist.

3 Taylor & Maclaurin Polynomials

Before looking at infinite series, we first develop a class of polynomials useful in the approximation of functions: Taylor polynomials. The simplest case of a Taylor polynomial is something you have seen already when you studied linear approximation, so we'll begin by reviewing that topic and extending it with some new theory.

3.1 Linear Approximation

Recall the ***linear approximation*** (or ***tangent line approximation***) to a function f at a point $x = a$: If f is differentiable at a, then for x near a

$$f(x) \approx f(a) + f'(a)(x - a) \ .$$

Here $y = f(a) + f'(a)(x - a)$ is just the equation of the tangent line to $y = f(x)$ at the point where $x = a$, so linear approximation is simply the idea of using the tangent line to approximate function values.

Letting $T_1(x) = f(a) + f'(a)(x - a)$, the function describing the tangent line, notice that

$$T_1(a) = f(a) \quad \text{and} \quad T_1'(a) = f'(a) \ .$$

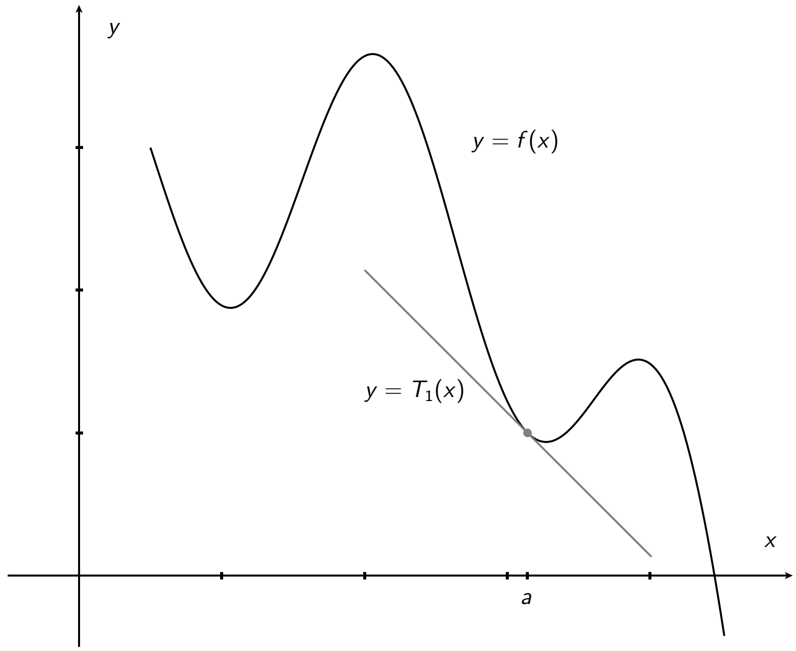

Figure 1: $f(x)$, $T_1(x)$

That is, the relatively simple function $T_1(x)$ agrees with $f(x)$ in function value and first derivative at $x = a$. The function $T_1(x)$ is called the ***Taylor polynomial of degree 1 for*** f ***at*** a. We see from plots of $y = f(x)$ and $y = T_1(x)$ in Figure 1 that for x near a the vertical distance between points $(x, f(x))$ and $(x, T_1(x))$ is small, but that this distance increases as x moves away from a.

We would like to say something more precise about how good the approximation $f(x) \approx T_1(x)$ is. Let $R_1(x)$ be the difference between $f(x)$ and $T_1(x)$:

$$R_1(x) = f(x) - [f(a) + f'(a)(x - a)] \ .$$

Think of $R_1(x)$ as the error in the approximation $f(x) \approx f(a) + f'(a)(x - a)$. How large can $R_1(x)$ possibly be? The answer is given by the following theorem.

Theorem 1

Suppose that f is twice differentiable on an open interval $\mathcal{I}$ containing containing a. Then for each x in $\mathcal{I}$

$$R_1(x) = \frac{f''(z)}{2}(x - a)^2$$

for some z between a and x.

Proof

The proof uses Rolle's theorem applied to a specially constructed function. Suppose a and x are in $\mathcal{I}$, $x \neq a$, and consider x fixed (that is, a constant). Define

$$g(t) = f(x) - f(t) - f'(t)(x - t) - R_1(x)\frac{(x - t)^2}{(x - a)^2} \ .$$

Notice that $g(a) = g(x) = 0$, so by Rolle's Theorem there is some number z strictly between a and x such that $g'(z) = 0$. Now

$$\begin{aligned} g'(t) &= 0 - f'(t) - f''(t)(x-t) + f'(t) + 2R_1(x)\frac{(x-t)}{(x-a)^2} \\ &= -f''(t)(x-t) + 2R_1(x)\frac{(x-t)}{(x-a)^2} \end{aligned}$$

so

$$g'(z) = -f''(z)(x-z) + 2R_1(x)\frac{(x-z)}{(x-a)^2} = 0$$

giving

$$R_1(x) = \frac{f''(z)}{2}(x-a)^2$$

□

Notice here that z is some (unknown) number between a and x, so we cannot in general give the exact value of $R_1(x)$. However, by analyzing f'' we can establish an ***upper bound*** on $|f''(z)|$ (that is, a greatest possible size), and thereby give a bound on the error term $|R_1(x)|$ itself. This is illustrated in the following example.

Example 1

Let $f(x) = x + \ln x$.

(i) Find the linear approximation to f at $a = 1$.

(ii) Use the linear approximation to approximate $f(1.1)$.

(iii) Give an error bound for your approximation.

Solution

(i) Here

$$f(x) = x + \ln x \;\text{ and }\; f'(x) = 1 + \frac{1}{x}\,.$$

Evaluating these at $a = 1$ gives

$$f(1) = 1 \;\text{ and }\; f'(1) = 2,$$

so we have

$$\begin{aligned} T_1(x) &= f(a) + f'(a)(x-a) \\ &= 1 + 2(x-1) \end{aligned}$$

(ii) The linear approximation is $f(1.1) \approx T_1(1.1) = 1 + 2(1.1 - 1) = 1.2$

(iii) We require an upper bound on $|R_1(1.1)|$. By Theorem 1

$$R_1(1.1) = \frac{f''(z)}{2}(1.1-1)^2$$

for some $1 < z < 1.1$. Here $f''(z) = -1/z^2$, so

$$|R_1(1.1)| = \left| \frac{-1}{2z^2}(1.1-1)^2 \right| .$$

Now ask: what is the largest possible size of $|R_1(1.1)|$ for $1 < z < 1.1$? In this case, taking z as small as possible makes $|R_1(1.1)|$ as large as possible, so setting $z = 1$ we have

$$|R_1(1.1)| \leq \left| \frac{-1}{2(1)^2}(1.1-1)^2 \right| = \frac{1}{200} .$$

That is,

$$|f(1.1) - T_1(1.1)| \leq \frac{1}{200}$$

□

This example shows that $T(1.1)$ is within $1/200$ or 0.005 of the true value of $f(1.1)$. A calculator check shows that this is indeed the case:

$$|f(1.1) - T(1.1)| = |(1.1 + \ln(1.1)) - (1 + 2(1.1-1))| \doteq 0.0047 ,$$

a remarkably accurate result given the simple form of the approximating function $T_1(x)$. In the sections to follow we will see how to improve the accuracy even further by extending $T_1(x)$ to a more general polynomial.

3.2 Taylor & Maclaurin Polynomials

Let n be a positive integer and suppose that f is n-times differentiable at $x = a$. Let's now generalize the linear approximation idea by finding a polynomial $T_n(x)$ of degree n which agrees with $f(x)$ in function value and first n-derivatives at $x = a$, i.e.

$$\begin{aligned} T_n(a) &= f(a), \\ T_n'(a) &= f'(a), \\ T_n''(a) &= f''(a), \\ &\vdots \\ T_n^{(n)}(a) &= f^{(n)}(a) \end{aligned}$$

Since T_n has degree n it can be expressed in the general form

$$T_n(x) = b_0 + b_1(x-a) + b_2(x-a)^2 + \cdots + b_n(x-a)^n$$

for some constants $b_0, b_1, \ldots, b_n$. To determine the constants $b_0, b_1, \ldots, b_n$ proceed as follows:

- Since $T_n(a) = f(a)$ we must have
$$f(a) = b_0 + b_1(a-a) + b_2(a-a)^2 + \cdots + b_n(a-a)^n$$
so that $b_0 = f(a)$.
- Since $T_n'(a) = f'(a)$ we must have
$$f'(a) = b_1 + 2b_2(a-a) + \cdots + nb_n(a-a)^{n-1}$$
so that $b_1 = f'(a)$.
- Since $T_n''(a) = f''(a)$ we must have
$$f''(a) = 2b_2 + (3)(2)b_3(a-a) + \cdots + n(n-1)b_n(a-a)^{n-2}$$
so that $b_2 = \dfrac{f''(a)}{2}$.
- Continuing in this way we find in general
$$b_k = \frac{f^{(k)}(a)}{k!}, k = 0, 1, \ldots, n$$

We can now define the ***Taylor polynomial*** of degree n for f at a:

Definition 3

If f is n-times differentiable at a, the ***Taylor polynomial*** of degree n for f at a is
$$T_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$
If $a = 0$, $T_n(x)$ is instead called a ***Maclaurin polynomial*** of degree n for f.

Some comments about this definition:

(i) Here '!' is ***factorial notation***. $n!$ is read "n factorial", so 5! is read "five-factorial". The factorial function is defined on the non-negative integers as follows:
$$\begin{aligned} 0! &= 1 \\ 1! &= 1 \\ 2! &= (1)(2) \\ 3! &= (1)(2)(3) \\ &\vdots \\ k! &= (1)(2)\cdots(k) \end{aligned}$$
This can also be defined recursively as $0! = 1$ and $k! = k(k-1)!$ for $k \geq 1$.

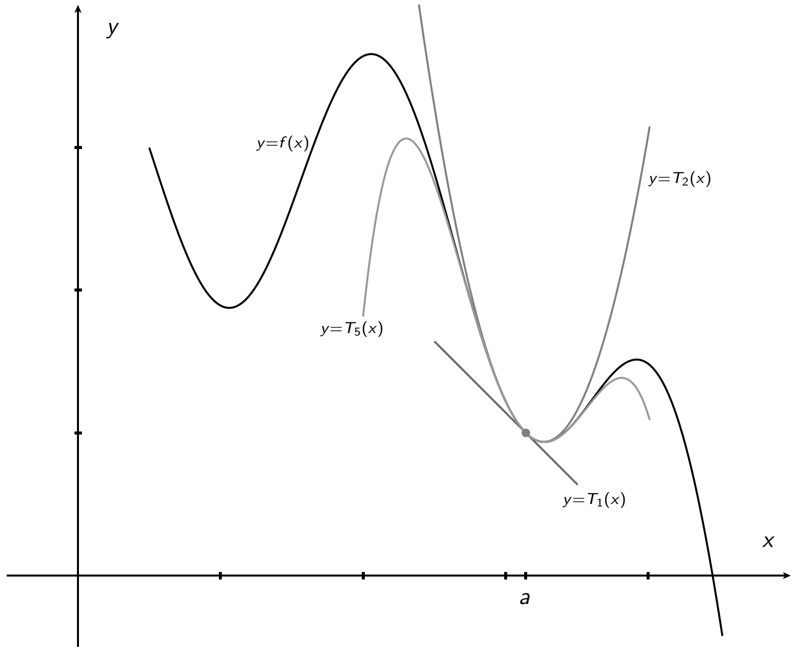

Figure 2: $f(x)$, $T_1(x)$, $T_2(x)$, $T_5(x)$

(ii) Notice that the first two terms of $T_n(x)$ give $T_1(x)$, the first three terms give $T_2(x)$, and so on. In other words, once you find $T_n(x)$ you also have $T_k(x)$ for $k = 1, 2, \ldots, n-1$.

Since $T_n(x)$ agrees with $f(x)$ in function value and first n-derivatives at $x = a$ we expect $f(x) \approx T_n(x)$ for x near a. See Figure 2 for an illustration of the improved approximations given by $T_1(x)$, $T_2(x)$ and $T_5(x)$.

3.3 Taylor's Theorem

As with linear approximation, when using a Taylor polynomial to approximate a function we would like an estimate of the error involved. There are several methods for doing this, one of which is a generalization of the error estimate we found for linear approximation. To state the result let

$$R_n(x) = f(x) - T_n(x)$$

be the error in the approximation.

Theorem 2

Suppose f is $(n+1)$-times differentiable on an open interval $\mathcal{I}$ containing containing a. Then for each x in $\mathcal{I}$

$$R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}(x-a)^{n+1}$$

for some z between a and x.

Proof

The proof is nearly identical to the $n = 1$ case we saw before. Suppose a and x are in $\mathcal{I}$, $x \neq a$, and consider x fixed. Define

$$g(t) = f(x) - f(t) - f'(t)(x-t) - \frac{f''(t)}{2!}(x-t)^2 - \cdots - \frac{f^{(n)}(t)}{n!}(x-t)^n - R_n(x)\frac{(x-t)^{(n+1)}}{(x-a)^{(n+1)}}$$

and notice that $g(a) = g(x) = 0$. By Rolle's Theorem there is some number z strictly between a and x such that $g'(z) = 0$. Now differentiating and simplifying we find

$$g'(t) = -\frac{f^{(n+1)}(t)}{n!}(x-t)^n + (n+1)R_n(x)\frac{(x-t)^n}{(x-a)^{(n+1)}}$$

so that

$$g'(z) = -\frac{f^{(n+1)}(z)}{n!}(x-z)^n + (n+1)R_n(x)\frac{(x-z)^n}{(x-a)^{(n+1)}} \; .$$

Setting $g'(z) = 0$ and isolating $R_n(x)$ then gives

$$R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}(x-a)^{n+1} \; .$$

□

Putting all this together we have the formal statement of Taylor's Theorem, also known as ***Taylor's Formula***:

Taylor's Theorem

Suppose f is $(n+1)$-times differentiable on an open interval $\mathcal{I}$ containing containing a. Then for each x in $\mathcal{I}$

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \frac{f^{(n+1)}(z)}{(n+1)!}(x-a)^{n+1}$$

where z is some number between a and x.

Let's revisit our first example and improve the approximation:

Example 2

Let $f(x) = x + \ln x$.

(i) Find $T_3(x)$, the Taylor polynomial of degree 3 for f at $a = 1$.

(ii) Use $T_3(x)$ to approximate $f(1.1)$.

(iii) Give an error bound for your approximation.

Solution

(i) This time

$$\begin{aligned} f(x) &= x + \ln x\ , \\ f'(x) &= 1 + \frac{1}{x}\ , \\ f''(x) &= \frac{-1}{x^2} \\ \text{and}\ \ f'''(x) &= \frac{2}{x^3}. \end{aligned}$$

Evaluating these at $a = 1$ gives

$$f(1) = 1,\quad f'(1) = 2,\quad f''(1) = -1\ \text{ and }\ f'''(1) = 2$$

so we have

$$\begin{aligned} T_3(x) &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 \\ &= 1 + 2(x-1) - \frac{1}{2!}(x-1)^2 + \frac{2}{3!}(x-1)^3 \\ &= 1 + 2(x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 \end{aligned}$$

(ii) The approximation is

$$\begin{aligned} f(1.1) &\approx T_3(1.1) \\ &= 1 + 2(1.1-1) - \frac{1}{2}(1.1-1)^2 + \frac{1}{3}(1.1-1)^3 \\ &= 1 + \frac{1}{5} - \frac{1}{200} + \frac{1}{3000} \\ &= \frac{1793}{1500} \end{aligned}$$

(iii) We require a bound on $|R_3(1.1)|$. By Theorem 2

$$R_3(1.1) = \frac{f^{(4)}(z)}{4!}(1.1-1)^4$$

for some $1 < z < 1.1$. Here $f^{(4)}(z) = -6/z^4$, so

$$|R_4(1.1)| = \left|\frac{-6}{4!z^4}(1.1-1)^4\right| = \left|\frac{-1}{4z^4}(0.1)^4\right|\ .$$

Now ask: what is the largest possible size of $|R_3(1.1)|$ for $1 < z < 1.1$? Again in this case, taking z as small as possible makes $|R_3(1.1)|$ as large as possible, so setting $z = 1$ we have

$$|R_3(1.1)| \le \left| \frac{-1}{4(1)^4}(0.1)^4 \right| = \frac{1}{40{,}000} = 0.000\,025 \ .$$

That is,

$$|f(1.1) - T_3(1.1)| \le 0.000\,025 \ .$$

□

Section 3 Exercises

1. (a) Determine $T_1(x)$ for $f(x) = \sqrt{x^2+9}$ at $a = -4$ and use it to approximate $f(-3.9)$.
   (b) Give an error bound for your approximation in part (a)
2. Use a linear approximation $T_1(x)$ for $f(x) = x/(x+1)$ to approximate $f(1.3)$.
3. Find $T_1(x)$ and $T_2(x)$ for $f(x) = (1+x)^k$ at $a = 0$ (k is a constant here).
4. Use $T_2(x)$ from the previous exercise to approximate $\sqrt[3]{1.009}$ and give an error bound for your approximation.
5. Find the Taylor polynomial of degree 3 for $f(x) = \tan x$ at $a = \pi/4$.
6. Find the Maclaurin polynomial of degree 3 for $f(x) = xe^{-x}$.
7. Find the Maclaurin polynomials of degree 1, 2, 3 and 4 for $f(x) = 2 + 3x - 5x^2 - 7x^3 + 11x^4$. What do you notice?
8. Find the Taylor polynomial of degree 3 for $f(x) = 3 + x + 4x^2 - 2x^3$ at $a = 2$, and then expand and simplify your result. What do you notice?
9. (a) Find the Maclaurin polynomial of degree 5 for $f(x) = \sin x$.
   (b) Determine the value of b such that your approximation in part (a) is accurate to within 0.000 05 for x in the interval $(-b, b)$. [This one is tricky: here $T_5(x) = T_6(x)$ (why?) so b can be determined by analyzing either $R_5(x)$ or $R_6(x)$, each resulting in a different b value.]
10. Find the Maclaurin polynomial of degree 4 for $f(x) = e^x$. Based on your work, what would be the Maclaurin polynomial of degree n?

Answers

1. (a) $T_1(x) = 5 - \dfrac{4(x+4)}{5}$; $f(-3.9) \approx T_1(-3.9) = \dfrac{123}{25}$
   (b) $|R_1(-3.9)| \le \dfrac{9}{200[(3.9)^2+9]^{3/2}} \approx 0.0004$
2. $T_1(x) = \dfrac{x+1}{4}$; $f(1.3) \approx T_1(1.3) = \dfrac{23}{40}$
3. $T_1(x) = 1 + kx$; $T_2(x) = 1 + kx + \dfrac{k(k-1)}{2}x^2$
4. $T_2(0.009) = \dfrac{1{,}002{,}991}{1{,}000{,}000}$; $|R_2(0.009)| \le \dfrac{9}{2 \cdot 10^8} = 4.5 \times 10^{-8}$
5. $T_3(x) = 1 + 2\left(x - \frac{\pi}{4}\right) + 2\left(x - \frac{\pi}{4}\right)^2 + \frac{8}{3}\left(x - \frac{\pi}{4}\right)^3$
6. $T_3(x) = x - x^2 + \dfrac{x^3}{2}$
7. $T_1(x) = 2 + 3x$; $T_2(x) = 2 + 3x - 5x^2$; $T_3(x) = 2 + 3x - 5x^2 - 7x^3$; $T_4(x) = 2 + 3x - 5x^2 - 7x^3 + 11x^4$
8. $T_3(x) = 5 - 7(x-2) - 8(x-2)^2 - 2(x-2)^3$; $T_3(x) = f(x)$
9. (a) $T_5(x) = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!}$
   (b) If analyzing $R_5(x)$, $b = (9/250)^{\frac{1}{6}}$. If analyzing $R_6(x)$, $b = (63/250)^{\frac{1}{7}}$.
10. $T_4(x) = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \dfrac{x^4}{4!}$;
    $T_n(x) = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \dfrac{x^4}{4!} + \cdots + \dfrac{x^n}{n!}$

4 Taylor & Maclaurin Series

In the previous section we saw that if f has $n+1$ derivatives on an open interval $\mathcal{I}$ containing a then

$$f(x) = T_n(x) + R_n(x)$$

for every x in $\mathcal{I}$. This brings up a natural question: as n increases to ∞, can $f(x)$ be somehow expressed as a polynomial of "infinite degree". The answer is, yes, but with conditions.

Definition 4

If f has derivatives of all orders at a then the ***infinite series*** (or simply ***series***)

$$\begin{aligned} T(x) &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \frac{f^{(4)}(a)}{4!}(x-a)^4 + \cdots \\ &= \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k \end{aligned}$$

is called ***the Taylor series for f at a***. In the case where $a = 0$ this series is called the ***Maclaurin series for f***.[1]

So if a function has derivatives of all orders at a then we can define its Taylor series $T(x)$, but is $f(x) = T(x)$? The answer is given by the following theorem:

Theorem 3

Suppose f has derivatives of all orders on an open interval $\mathcal{I}$ containing a and that for each x in $\mathcal{I}$

$$\lim_{n\to\infty} R_n(x) = 0\ .$$

Then for each x in $\mathcal{I}$ we have

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

and we say that the Taylor (or Maclaurin) series $T(x)$ ***converges*** to $f(x)$ on $\mathcal{I}$. The largest open interval $\mathcal{I}$ containing a on which $T(x)$ converges to $f(x)$ is called the ***open interval of convergence*** of the series.

To be precise, when $f(x)$ is equal to it's Taylor series on an interval $\mathcal{I}$ and we write

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

[1] Note the introduction of ***sigma notation*** $\sum$ here which is convenient for representing the sum of a large number of terms, particularly when the terms follow a pattern. See the appendix in the back of *Essential Calculus, Early Transcendentals by James Stewart (1^{st} or 2^{nd} edition)* for a review of sigma notation.

we mean $f(x) = \lim_{n\to\infty} T_n(x)$ on $\mathcal{I}$

A measure of caution is necessary when working with Taylor (or Maclaurin) series: although a function f may have derivatives of all orders at a, so that the Taylor series $T(x)$ is defined, it does not necessarily follow that $f(x) = T(x)$ when $x \neq a$. Furthermore, it is possible that a function is equal to its Taylor series on an interval but not on the entire domain of f. The analysis of $R_n(x)$ as $n \to \infty$ is critical in the determination of when a function is equal to its Taylor (or Maclaurin) series.

It turns out that many familiar functions are equal to their Taylor (or Maclaurin) series. Before looking at these we first prove a limit result which often arises in the analysis of $R_n(x)$:

Theorem 4

For each real number x,

$$\lim_{n\to\infty} \frac{|x|^n}{n!} = 0$$

Proof

Let x be any fixed real number. If $0 \leq |x| < 1$ then

$$\frac{|x|^n}{n!} \leq \frac{1}{n!} \to 0 \text{ as } n \to \infty .$$

If $|x| \geq 1$, let m be the unique positive integer such that $m - 1 \leq |x| < m$. Then

$$\begin{aligned} \frac{|x|^n}{n!} &= \frac{|x|}{1} \cdot \frac{|x|}{2} \cdots \frac{|x|}{n} \\ &= \frac{|x|}{1} \cdots \frac{|x|}{m-1} \left(\frac{|x|}{m} \cdot \frac{|x|}{m+1} \cdots \cdot \frac{|x|}{n} \right) \\ &\leq \frac{|x|^{m-1}}{(m-1)!} \left(\frac{|x|}{m} \right)^{n-m+1} \\ &\to 0 \text{ as } n \to \infty \text{ since } \frac{|x|}{m} < 1 \end{aligned}$$

In either case, $\lim_{n\to\infty} \frac{|x|^n}{n!} = 0$. $\square$

As a first example, let's find the Maclaurin series for $f(x) = e^x$ and show that it converges to e^x for every real number x. Because this is such an important Maclaurin series we state the result as a theorem:

Theorem 5

For every real number x,

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

Proof

Every derivative of e^x is again e^x, so with $a = 0$ we have $f(0) = f'(0) = f''(0) = \cdots = f^{(n)}(0) = 1$. The resulting Maclaurin series for e^x is

$$\begin{aligned} T(x) &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \cdots \\ &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \\ &= \sum_{k=0}^{\infty} \frac{x^k}{k!} . \end{aligned}$$

To show that this Maclaurin series is equal to e^x for every real number x, consider the error associated with the Maclaurin polynomial of degree n:

$$R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}x^{n+1} = \frac{e^z}{(n+1)!}x^{n+1} \text{ where } z \text{ is some number between 0 and } x.$$

We must show that for each x, $R_n(x) \to 0$. If $x = 0$ then $R_n(x) = 0$ and we're done. If $x \neq 0$,

$$\begin{aligned} |R_n(x)| &= \left| \frac{e^z x^{n+1}}{(n+1)!} \right| \\ &= \frac{e^z |x|^{n+1}}{(n+1)!} \quad \text{since } e^z > 0 \\ &\le \frac{e^{|z|} |x|^{n+1}}{(n+1)!} \quad \text{since } z \le |z| \\ &\le \frac{e^{|x|} |x|^{n+1}}{(n+1)!} \quad \text{since } |z| < |x| \\ &\to 0 \quad \text{as } n \to \infty \quad \text{by Theorem 4.} \end{aligned}$$

Since $|R_n(x)| \to 0$ so does $R_n(x)$, and so by Theorem 3

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

□

Showing that $R_n(x) \to 0$ is not always so easy and there are other methods for showing that a function is equal to its Taylor or Maclaurin series. Here is a second example, this time for a function which is not equal to it's Maclaurin series on its entire domain, and we'll use a different method to analyze $R_n(x)$:

Example 3

Find the Maclaurin series for $f(x) = \dfrac{1}{1-x}$ and determine the open interval of convergence.

Solution

First note that the domain of f is all real numbers excluding $x = 1$. Evaluating f and its derivatives at $a = 0$ we find

$$f(x) = \frac{1}{(1-x)} \quad ; \quad f(0) = 1$$

$$f'(x) = \frac{1}{(1-x)^2} \quad ; \quad f'(0) = 1$$

$$f''(x) = \frac{2}{(1-x)^3} \quad ; \quad f''(0) = 2$$

$$f'''(x) = \frac{3!}{(1-x)^3} \quad ; \quad f'''(0) = 3!$$

$$f^{(4)}(x) = \frac{4!}{(1-x)^4} \quad ; \quad f^{(4)}(0) = 4!$$

and in general,

$$f^{(k)}(x) = \frac{k!}{(1-x)^k} \quad ; \quad f^{(k)}(0) = k!\ .$$

Inserting this into the formula for the Maclaurin series we find

$$\begin{aligned} T(x) &= 1 + 1 \cdot x + \frac{2}{2}x^2 + \frac{3!}{3!}x^3 - \frac{4!}{4!}x^4 + \cdots \\ &= 1 + x + x^2 + x^3 + x^4 + \cdots \\ &= \sum_{k=0}^{\infty} x^k\ . \end{aligned}$$

To analyze $R_n(x)$ we'll use a method that is unique to this example. For any x in the domain of f, (i.e. $x \neq 1$),

$$T_n(x) = 1 + x + x^2 + x^3 + x^4 + \cdots + x^n\ .$$

Multiplying both sides of this expression by x gives

$$xT_n(x) = x + x^2 + x^3 + x^4 + \cdots + x^{n+1}\ .$$

Substracting the second equation from the first then gives

$$\begin{aligned} T_n(x) - xT_n(x) = 1 &+ x + x^2 + x^3 + x^4 + \cdots + x^n \\ &- x - x^2 - x^3 - x^4 - \cdots - x^n - x^{n+1} \; . \end{aligned}$$

Factoring on the left and cancelling on the right results in

$$(1-x)T_n(x) = 1 - x^{n+1} \; ,$$

and isolating $T_n(x)$ then gives

$$T_n(x) = \frac{1-x^{n+1}}{1-x} \; .$$

The error term can then be expressed

$$R_n(x) = f(x) - T_n(x) = \frac{1}{1-x} - \frac{1-x^{n+1}}{1-x} = \frac{x^{n+1}}{1-x} \; .$$

Notice here that $R_n(x)$ is the exact error on the domain of f and does not depend on some unknown z between 0 and x. Furthermore, we see that $R_n(x) \to 0$ as $n \to \infty$ if and only if $|x| < 1$, i.e. $-1 < x < 1$. Therefore,

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k \text{ for } x \text{ in } (-1, 1) \; .$$

$\square$

The series

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k \text{ for } x \text{ in } (-1, 1)$$

from the previous example is called the ***geometric series*** and it is useful for constructing new Taylor series from existing ones as we'll see in the next section. Other standard series are likewise useful in this respect so we list them in a table of standard Maclaurin series at the end of this section.

Another important observation about Taylor (and Maclaurin) series is that they contain the Taylor (*resp.* Maclaurin) polynomials of every degree for the function in question. For example, using the fact that

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \quad \text{on } (-1, 1)$$

we can immediately conclude that, for example, the Maclaurin polynomial of degree 5 for $f(x) = \ln(1+x)$ is

$$T_5(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} \; .$$

Maclaurin Series	Open Interval of Convergence
$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$	$(-\infty, \infty)$
$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$	$(-\infty, \infty)$
$\cos x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$	$(-\infty, \infty)$
$\ln(1+x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{k+1}}{k+1} = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$	$(-1, 1)$
$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + x^3 + \cdots$	$(-1, 1)$
$\arctan x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2k+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots$	$(-1, 1)$

Section 4 Exercises

1. Show that $\sin x$ is equal to its Maclaurin series on $(-\infty, \infty)$.

2. Find the Maclaurin series for $f(x) = 2+3x-5x^2-7x^3+11x^4$ and state the open interval of convergence. This example shows that a Maclaurin (or Taylor) series need not have an infinite number of non-zero terms.

3. Find the Maclaurin series for $f(x) = xe^x$.

4. Find the Taylor series for $f(x) = 3+x+4x^2-2x^3$ at $a = 2$ and state the open interval of convergence.

5. Find the Maclaurin series for $f(x) = \dfrac{x^2}{x+1}$ (Hint: to make the calculation of derivatives easier, use long division of polynomials to first express $f(x)$ in the form
$$f(x) = ax + b + \frac{c}{x+1}$$
for suitable a, b and c.)

6. Find the Taylor series at $a = 1$ for $f(x) = \dfrac{1}{x^2}$.

7. What is the sum of the infinite series
$$1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots \quad ?$$

8. What is the sum of the infinite series
$$\pi - \frac{\pi^3}{3!} + \frac{\pi^5}{5!} + \cdots \quad ?$$

9. The Maclaurin series for $f(x) = x^3 e^{-x^2}$ is
$$T(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+3}}{k!} \ .$$
Determine $f^{(13)}(0)$ and $f^{(14)}(0)$.

10. Use the fact that
$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k \quad \text{on } (-1,1)$$
to find a Maclaurin series which is equal to $g(x) = \dfrac{1}{1+x^2}$. What is the open interval of convergence?

Answers

2. $T(x) = 2 + 3x - 5x^2 - 7x^3 + 11x^4$; $(-\infty, \infty)$

3. $T(x) = \sum_{k=0}^{\infty} \frac{x^{k+1}}{k!}$

4. $T(x) = 5 - 7(x-2) - 8(x-2)^2 - 2(x-2)^3$; $(-\infty, \infty)$

5. $T(x) = \sum_{k=0}^{\infty} (-1)^k x^{k+2}$

6. $T(x) = \sum_{k=0}^{\infty} (-1)^k (k+1)(x-1)^k$

7. $e^1 = e$

8. $\sin \pi = 0$

9. $f^{(13)}(0) = -\frac{13!}{5!} = 51{,}891{,}840$ and $f^{(14)}(0) = 0$

10. $T(x) = \sum_{k=0}^{\infty} (-1)^k x^{2k}$; $\mathcal{I} = (-1, 1)$

5 Working with Taylor & Maclaurin Series

Functions that are equal to their Taylor (*resp.* Maclaurin) series are equal to the limit as $n \to \infty$ of their Taylor (*resp.* Maclaurin) polynomials. This fact, together with standard limit laws, allows us to construct Taylor and Maclaurin series for new functions by combining the Taylor and Maclaurin series of known functions in much the same way one would treat polynomials.

Theorem 6

Suppose
$$f(x) = \sum_{k=0}^{\infty} b_k (x-a)^k$$
$$\text{and} \quad g(x) = \sum_{k=0}^{\infty} c_k (x-a)^k$$
are Taylor series representations for f and g at the point a with open intervals of convergence $\mathcal{I}_1$ and $\mathcal{I}_2$, respectively. Then on $\mathcal{I} = \mathcal{I}_1 \cap \mathcal{I}_2$, (that is, $\mathcal{I}$ is the largest open interval contained in both $\mathcal{I}_1$ and $\mathcal{I}_2$),

(i) $(f+g)(x) = \sum_{k=0}^{\infty} p_k (x-a)^k$ where $p_k = b_k + c_k$

(ii) $(f-g)(x) = \sum_{k=0}^{\infty} p_k(x-a)^k$ where $p_k = b_k - c_k$

(iii) $(fg)(x) = \sum_{k=0}^{\infty} p_k(x-a)^k$ where $p_k = \sum_{j=0}^{k} b_j c_{k-j}$

For each of the new series in (i), (ii) and (iii), the open interval of convergence may be $\mathcal{I}$ itself or possibly a larger interval containing $\mathcal{I}$.

In simpler terms, this theorem tells us that the Taylor series for the sum, difference and product of two functions is found by simply adding, subtracting, and multiplying (respectively) the Taylor series for the two functions as one would do with polynomials. That this is true for sums and differences of functions is not that surprising. The fact that this is also true for products is worth a closer look. The reason that Theorem 6 is true has to do with a property of more general series we'll look at in the next section. For now we quote the the needed fact:

Theorem 7

Suppose the series

$$f(x) = r_0 + r_1(x-a) + r_2(x-a)^2 + r_3(x-a)^3 + \cdots = \sum_{k=0}^{\infty} r_k(x-a)^k$$

converges for every x in an open interval $\mathcal{I}$ containing a. Then f is differentiable on $\mathcal{I}$,

$$f'(x) = r_1 + 2r_2(x-a) + 3r_3(x-a)^2 + 4r_4(x-a)^3 + \cdots = \sum_{k=1}^{\infty} k r_k(x-a)^{k-1}$$

and this series for $f'(x)$ also converges on $\mathcal{I}$.

This theorem tells us that the Taylor (or Maclaurin) series for the derivative of a function is obtained by simply differentiating the Taylor (*resp.* Maclaurin) series of the original function term by term. We can now prove our main theorem:

Proof of Theorem 6

The proofs of (i), (ii) and (iii) are similar; let's prove (iii) since it's the least obvious of the three. Let $F_n(x)$ and $G_n(x)$ be the Taylor polynomials of degree n at a for f and g, respectively. By hypothesis

$$f(x) = \lim_{n\to\infty} F_n(x) \ \text{ for every } x \text{ in } \mathcal{I}_1$$

and

$$g(x) = \lim_{n\to\infty} G_n(x) \ \text{ for every } x \text{ in } \mathcal{I}_2 \ .$$

Since these limits exist and $\mathcal{I}$ is contained in both $\mathcal{I}_1$ and $\mathcal{I}_2$, then for every x in $\mathcal{I}$

$$\begin{aligned} f(x)g(x) &= \left(\lim_{n\to\infty} F_n(x)\right)\left(\lim_{n\to\infty} G_n(x)\right) \\ &= \lim_{n\to\infty} \left(F_n(x)G_n(x)\right) \\ &= \sum_{k=0}^{\infty} p_k(x-a)^k \end{aligned}$$

where $p_k = \displaystyle\sum_{j=0}^{k} b_j c_{k-j}$ are the coefficients resulting from multiplying the series together and collecting like terms.

So $(fg)(x)$ has a convergent series representation on $\mathcal{I}$; to complete the proof we must show that this series is indeed a Taylor series. That is, we must show that for each k,

$$p_k = \frac{1}{k!}\frac{d^k}{dx^k}\left[f(x)g(x)\right]_{x=a} .$$

Setting $x = a$ in

$$f(x)g(x) = p_0 + p_1(x-a) + p_2(x-a)^2 + p_3(x-a)^3 + p_4(x-a)^4 + \cdots$$

we find

$$p_0 = f(a)g(a) .$$

Taking k derivatives and setting $x = a$ in

$$f(x)g(x) = p_0 + p_1(x-a) + p_2(x-a)^2 + p_3(x-a)^3 + p_4(x-a)^4 + \cdots$$

gives (by Theorem 7)

$$\begin{aligned} \frac{d^k}{dx^k}\left[f(x)g(x)\right]_{x=a} &= \frac{d^k}{dx^k}\left[p_0 + p_1(x-a) + p_2(x-a)^2 + p_3(x-a)^3 + p_4(x-a)^4 + \cdots\right]_{x=a} \\ &= \left[\, k!p_k + (\text{terms with at least one factor of } (x-a))\,\right]_{x=a} \\ &= k!p_k \end{aligned}$$

so that

$$p_k = \frac{1}{k!}\frac{d^k}{dx^k}\left[f(x)g(x)\right]_{x=a} ,$$

which completes the proof. □

Here are some examples illustrating the application of these rules:

Example 4

Find the Maclaurin series for $f(x) = x^2e^x$.

Solution

$$
\begin{aligned}
x^2 e^x &= x^2 \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right) \\
&= x^2 + x^3 + \frac{x^4}{2!} + \frac{x^5}{3!} + \cdots \\
&= \sum_{k=0}^{\infty} \frac{x^{k+2}}{k!}
\end{aligned}
$$

Since the Maclaurin series for x^2 and e^x both converge on $(-\infty, \infty)$, the open interval of convergence for $x^2 e^x$ will again be $(-\infty, \infty)$. $\square$

Notice in this example that the final answer was stated using $\sum$ notation. This is not strictly necessary, but is good practice in the case where the terms of the sum follow an obvious pattern. You will generally be told the form to use for expressing your final answer.

Example 5

Find the first four non-zero terms of the Maclaurin series for $f(x) = e^x \cos x$.

Solution

Multiply each term from the first series by each term in the second and collect like terms:

$$
\begin{aligned}
e^x \cos x &= \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right) \\
&= 1 \cdot \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right) + x \cdot \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right) \\
&\quad + \frac{x^2}{2!} \cdot \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right) + \frac{x^3}{3!} \cdot \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right) \\
&\quad + \frac{x^4}{4!} \cdot \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right) + \cdots \\
&= 1 + x - \frac{x^3}{3} - \frac{x^4}{6} + \cdots
\end{aligned}
$$

Again in this example, the Maclaurin series for e^x and $\cos x$ both converge on $(-\infty, \infty)$ and so the open interval of convergence for the product is also $(-\infty, \infty)$. $\square$

Another way to define new functions from old is by function composition. Recall that $(f \circ g)(x) = f(g(x))$. In the case where $g(x) = x^r$ for r a non-negative integer we have

Theorem 8

If
$$f(x) = \sum_{k=0}^{\infty} b_k x^k$$
is a Maclaurin series for f with open interval of convergence $\mathcal{I}$ and r is a non-negative integer such that x^r is in $\mathcal{I}$, then
$$f(x^r) = \sum_{k=0}^{\infty} b_k x^{rk}$$
is the Maclaurin series for $f(x^r)$.

Example 6

Find the Maclaurin polynomial of degree 6 for $f(x) = \dfrac{x}{1+4x^2}$.

Solution

Let's find the Maclaurin series for f and then extract the terms up to degree 6. First rearrange $f(x)$ to make the choice of series more obvious:
$$f(x) = \frac{x}{1+4x^2} = x \cdot \frac{1}{1-(-4x^2)} \ .$$

For the fraction $\dfrac{1}{1-(-4x^2)}$ we'll use the geometric series for $\dfrac{1}{1-x}$ with x replaced with $-4x^2$:
$$\begin{aligned}
f(x) &= x \cdot \frac{1}{1-(-4x^2)} \\
&= x \cdot \left(1 + (-4x^2) + (-4x^2)^2 + (-4x^2)^3 + (-4x^2)^4 + \cdots\right) \\
&= x \cdot \left(1 - 4x^2 + 16x^4 - 64x^6 + \cdots\right) \\
&= x - 4x^3 + 16x^5 - 64x^7 + \cdots
\end{aligned}$$

Extracting the terms of degree 6 or less we have
$$T_6(x) = x - 4x^3 + 16x^5 \ .$$

$\square$

Taylor and Maclaurin series can be used to calculate limits, and in some cases this approach is far more efficient than L'Hospital's Rule as the following example illustrates:

Example 7

Evaluate $\lim\limits_{x\to 0} \dfrac{e^{-x^2}+x^2-1}{x^4}$

Solution

Applying direct substitution yields the indeterminate form "$\dfrac{0}{0}$". Let's replace e^{-x^2} with its Maclaurin series, which is easily found by replacing x with $-x^2$ in the series for e^x:

$$\begin{aligned}
\lim_{x\to 0} \frac{e^{-x^2}+x^2-1}{x^4} &= \lim_{x\to 0} \frac{\left(1+(-x^2)+\dfrac{(-x^2)^2}{2!}+\dfrac{(-x^2)^3}{3!}+\cdots\right)+x^2-1}{x^4} \\
&= \lim_{x\to 0} \frac{\left(1-x^2+\dfrac{x^4}{2!}-\dfrac{x^6}{3!}+\cdots\right)+x^2-1}{x^4} \\
&= \lim_{x\to 0} \frac{\left(\dfrac{x^4}{2!}-\dfrac{x^6}{3!}+\cdots\right)}{x^4} \\
&= \lim_{x\to 0} \frac{x^4\left(\dfrac{1}{2!}-\dfrac{x^2}{3!}+\cdots\right)}{x^4} \\
&= \lim_{x\to 0} \frac{1}{2!}-\frac{x^2}{3!}+\cdots \\
&= \frac{1}{2}
\end{aligned}$$

□

Section 5 Exercises

1. Find the Maclaurin series and state the open interval of convergence:

   (a) $f(x)=\cos(\pi x)$

   (b) $f(x)=x\arctan x$

2. Find the first four non-zero terms of the Maclaurin series for $f(x)=e^x\ln(1-x)$.

3. Use a series to evaluate the following limits:

   (a) $\lim\limits_{x\to 0} \dfrac{1-\cos x}{1+x-e^x}$

(b) $\lim_{x\to 0} \dfrac{\sin x - x + \dfrac{x^3}{6}}{x^5}$

4. Find the Maclaurin polynomial of degree 13 for $f(x) = x^3 \sin(x^2)$.

5. Find the Maclaurin series for $f(x) = \dfrac{x}{(1+x)^2}$ (Hint: $\dfrac{d}{dx}\left[\dfrac{1}{1+x}\right] = \dfrac{-1}{(1+x)^2}$.)

6. Find the Maclaurin series and state the open interval of convergence for $f(x) = \sin^2 x$. (Hint: using a trigonometric identity to first rewrite $f(x)$ will make this easier.)

7. Find the Maclaurin series for $g(x) = 2\sin x \cos x$ (Hint: use your result from the previous exercise.)

8. The hyperbolic sine function, $\sinh x$, is defined to be

$$\sinh x = \frac{e^x - e^{-x}}{2} .$$

This function arises in the solution to certain equations in physics and engineering. Find the Maclaurin series for $\sinh(x^2/3)$.

9. We know that $\dfrac{d}{dx}[e^x] = e^x$. Prove this by differentiating the Maclaurin series for e^x .

10. Show that the derivative of $\sin x = \cos x$ using Maclaurin series.

Answers

1. (a) $\sum_{k=0}^{\infty} \dfrac{(-1)^k \pi^{2k} x^{2k}}{(2k)!}$; $(-\infty, \infty)$

   (b) $\sum_{k=0}^{\infty} \dfrac{(-1)^k x^{2(k+1)}}{2k+1}$; $(-1, 1)$

2. $-x - \dfrac{3x^2}{2} - \dfrac{4x^3}{3} - x^4 - \cdots$

3. (a) -1

   (b) $\dfrac{1}{120}$

4. $T_{13}(x) = x^5 - \dfrac{x^9}{6} + \dfrac{x^{13}}{120}$

5. $\sum_{k=0}^{\infty} (-1)^k (k+1) x^{k+1}$

6. $\sum_{k=0}^{\infty} \dfrac{(-1)^k 2^{2k+1} x^{2(k+1)}}{(2(k+1))!}$; $(-\infty, \infty)$

7. $\sum_{k=0}^{\infty} \dfrac{(-1)^k 2^{2k+1} x^{2k+1}}{(2k+1)!}$

8. $\sum_{k=0}^{\infty} \dfrac{x^{2(2k+1)}}{3^{2k+1}(2k+1)!}$

6 Power Series

So far we have seen that if a function f has derivatives of all orders at a point $x = a$ then the Taylor (or Maclaurin) series for the function can be defined, and under suitable conditions, the series converges to the function on an open interval of convergence $\mathcal{I}$. That is, for x in $\mathcal{I}$,

$$f(x) = \sum_{k=0}^{\infty} a_k (x-a)^k ,$$

where equality here is understood to mean

$$f(x) = \lim_{n\to\infty} \sum_{k=0}^{n} a_k (x-a)^k .$$

Now let's examine this idea from the other direction: suppose we have an infinite sequence of real numbers $a_0, a_1, a_2, \ldots$ and we form the infinite series

$$\sum_{k=0}^{\infty} a_k(x-a)^k \ .$$

Now let

$$f(x) = \sum_{k=0}^{\infty} a_k(x-a)^k = \lim_{n\to\infty} \sum_{k=0}^{n} a_k(x-a)^k \ ,$$

where the domain of f is the set of real numbers x for which the limit above exists. What can we say about functions defined in this way? In particular, what is the domain, is the function differentiable, and if so, what is the derivative?

Definition 5

A series of the form

$$\sum_{k=0}^{\infty} a_k(x-a)^k = a_0 + a_1(x-a) + a_2(x-a)^2 + a_3(x-a)^3 + \cdots$$

is called a ***power series centred at*** **a** or a ***power series about*** **a** . The real numbers $a_0, a_1, a_2, \ldots$ are called the ***coefficients*** of the series.

Notice that Taylor and Maclaurin series fall into the general category of power series. Also notice that $f(a) = a_0$ for every power series $f(x) = \sum_{k=0}^{\infty} a_k(x-a)^k$, so there is at least one point in the domain of f. To determine the open interval of convergence of the power series, we turn to the following theorem which is a consequence of a result called the ***ratio test*** from the general theory of infinite series:

Theorem 9

Suppose $\sum_{k=0}^{\infty} a_k(x-a)^k$ is a power series and let $u_k(x)$ be the k^{th} non-zero term of the series (note that some of the a_k may be zero). Then the power series converges if

$$\lim_{k\to\infty} \frac{|u_{k+1}(x)|}{|u_k(x)|} < 1$$

We illustrate the use of the theorem with two examples.

Example 8

Determine the open interval of convergence of

$$\sum_{k=0}^{\infty} \frac{(-1)^k x^{k+1}}{k+1}$$

Solution

Here $u_k(x) = \dfrac{(-1)^k x^{k+1}}{k+1}$. For convergence we require

$$\lim_{k\to\infty} \frac{|u_{k+1}(x)|}{|u_k(x)|} < 1$$

$$\lim_{k\to\infty} \frac{\left|\dfrac{(-1)^{k+1}x^{k+2}}{k+2}\right|}{\left|\dfrac{(-1)^k x^{k+1}}{k+1}\right|} < 1$$

$$\lim_{k\to\infty} \left|\frac{x^{k+2}}{x^{k+1}} \frac{k+1}{k+2}\right| < 1$$

$$\lim_{k\to\infty} \left|x \frac{k+1}{k+2}\right| < 1$$

$$|x| < 1$$

So the open interval of convergence is $-1 < x < 1$, i.e. $\mathcal{I} = (-1, 1)$.

You may notice from our work on Taylor series that the power series in this example is the Taylor series defiining $\ln(1+x)$, and $\mathcal{I} = (-1, 1)$ is the result stated in the summary of Maclaurin series at the end of Section 4. □

Example 9

Determine the open interval of convergence of $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k 4^k (x-1)^{2k}}{k}$

Solution

Here

$$u_k(x) = \frac{(-1)^k 4^k (x-1)^{2k}}{k}$$

For convergence of the series we require

$$\lim_{k\to\infty} \frac{|u_{k+1}(x)|}{|u_k(x)|} < 1$$

$$\lim_{k\to\infty} \frac{\left|\dfrac{(-1)^{k+1}4^{k+1}(x-1)^{2(k+1)}}{k+1}\right|}{\left|\dfrac{(-1)^k 4^k (x-1)^{2k}}{k}\right|} < 1$$

$$\lim_{k\to\infty} \left|\frac{4^{k+1}}{4^k}\frac{(x-1)^{2k+2}}{(x-1)^{2k}}\frac{k}{k+1}\right| < 1$$

$$\lim_{k\to\infty} \left|4(x-1)^2\frac{k}{k+1}\right| < 1$$

$$4|x-1|^2 < 1$$

$$|x-1|^2 < \frac{1}{4}$$

$$\text{so } \ |x-1| < \frac{1}{2}$$

That is, the series converges for $1/2 < x < 3/2$, so the open interval of convergence is $\mathcal{I} = (1/2, 3/2)$. □

In these two examples, the limit

$$\lim_{k\to\infty} \frac{|u_{k+1}(x)|}{|u_k(x)|} < 1$$

resulted in a statement of the form $|x-a| < R$ which gave the open interval of convergence $(a-R, a+R)$. The value R here is called the ***radius of convergence*** of the power series. There are three possibilities for R:

(i) If $R = 0$, the power series converges only for $x = a$ and the open interval of convergence does not exist.

(ii) If $R > 0$, the power series converges for $|x-a| < R$ and diverges for $|x-a| > R$. That is, the open interval of convergence of the power series is $(a-R, a+R)$.

(iii) If $R = \infty$ the power series converges for all real numbers: the open interval of convergence is $(-\infty, \infty)$.

The first two examples above show possibility (ii) where $R > 0$ is a finite number. The next two examples show the other two possibilities: $R = 0$ and $R = \infty$:

Example 10 $(R = 0)$

Determine the radius of convergence and open interval of convergence of

$$\sum_{k=0}^{\infty} k!(x-2)^k$$

Solution

Here $u_k(x) = k!(x-2)^k$. We require

$$\lim_{k\to\infty} \frac{|u_{k+1}(x)|}{|u_k(x)|} < 1$$

$$\lim_{k\to\infty} \frac{|(k+1)!(x-2)^{k+1}|}{|k!(x-2)^k|} < 1$$

$$\lim_{k\to\infty} |(k+1)(x-2)| < 1$$

which is only possible if $x = 2$, and so $R = 0$. In this case the open interval of convergence does not exist. □

Example 11 $(R = \infty)$

Determine the radius of convergence and open interval of convergence of

$$\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{2^{2k}(k!)^2}$$

Solution

Here $u_k = \dfrac{(-1)^k x^{2k}}{2^{2k}(k!)^2}$ and we require

$$\lim_{k\to\infty} \frac{|u_{k+1}(x)|}{|u_k(x)|} < 1$$

$$\lim_{k\to\infty} \frac{\left|\dfrac{(-1)^{k+2}x^{2(k+1)}}{2^{2(k+1)}((k+1)!)^2}\right|}{\left|\dfrac{(-1)^k x^{2k}}{2^{2k}(k!)^2}\right|} < 1$$

$$\lim_{k\to\infty} \left| \frac{2^{2k}}{2^{2k+2}} \frac{k!}{(k+1)!} \frac{k!}{(k+1)!} \frac{x^{2k+2}}{x^{2k}} \right| < 1$$

$$\lim_{k\to\infty} \left| \frac{1}{4} \frac{1}{(k+1)^2} x^2 \right| < 1$$

$$\text{which results in } \ 0 < 1,$$

and this last statement is true for every real number x. So $R = \infty$ in this case and the open interval of covergence is the entire real number line, $\mathcal{I} = (-\infty, \infty)$.

(The function defined by the power series in this example is called a Bessel function of order 0. Bessel functions arise in the solutions of problems involving wave propagation, among others.) □

In the case where $R > 0$ and finite, the power series may also converge for $x = a - R$ or for $x = a + R$ (that is, at one or both of the boundary points of the open interval of convergence.) The convergence behaviour at each of these two points must be analyzed separately using other techniques from the general theory of infinite series. For our purposes, it will suffice to find the radius of convergence and, in the case of $R > 0$, the corresponding open interval of convergence.

Derivatives of Power Series

The derivatives of functions defined by power series can be found as one might expect, by differentiating term by term as one would a polynomial. The details are contained the in the following theorem which we state without proof:

Theorem 10

Suppose $\displaystyle\sum_{k=0}^{\infty} a_k(x-a)^k$ is a power series with radius of convergence $R > 0$ and corresponding open interval of convergence $\mathcal{I} = (a - R, a + R)$. Then the function

$$f(x) = \sum_{k=0}^{\infty} a_k(x-a)^k$$

is differentiable on $\mathcal{I}$,

$$f'(x) = a_1 + 2a_2(x-a) + 3a_3(x-a)^2 + 4a_4(x-a)^3 + \cdots$$
$$= \sum_{k=1}^{\infty} k a_k (x-a)^{k-1} \ ,$$

and the radius of convergence of the power series defining $f'(x)$ is again R.

As a consequence of this theorem, if $f(x) = \sum_{k=0}^{\infty} a_k(x-a)^k$ on $\mathcal{I}$, then

$$a_k = \frac{f^{(k)}(a)}{k!} \quad \text{for } k = 0, 1, 2, \ldots$$

In other words, the power series $\sum_{k=0}^{\infty} a_k(x-a)^k$ is the Taylor series about a for the function f it defines, and the open interval of convergence of the Taylor series is $\mathcal{I}$.

Example 12

The first five non-zero terms of the power series representing $\tan x$ on $\mathcal{I} = (-\pi/2, \pi/2)$ are

$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \frac{62x^9}{2835} + \cdots \ .$$

Use this information to find the Maclaurin series for $\sec^2 x$. State the open interval of convergence.

Solution

Since the given series represents $\tan x$ on $\mathcal{I}$, it is equal to the Maclaurin series for the function. Differentiating gives

$$\frac{d}{dx}[\tan x] = \frac{d}{dx}\left[x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \frac{62x^9}{2835} + \cdots\right]$$
$$\sec^2 x = 1 + \frac{3x^2}{3} + \frac{10x^4}{15} + \frac{119x^6}{315} + \frac{558x^8}{2835} + \cdots$$
$$= 1 + x^2 + \frac{2x^4}{3} + \frac{17x^6}{45} + \frac{62x^8}{315} + \cdots$$

The open interval of convergence is the same as that of the original series: $\mathcal{I} = (-\pi/2, \pi/2)$ $\square$

Section 6 Exercises

1. Find the radius of convergence and open interval of convergence:

   (a) $\displaystyle\sum_{k=0}^{\infty} \frac{(x-2)^k}{10^k}$

   (b) $\displaystyle\sum_{k=0}^{\infty} \frac{(3x-2)^{k+1}}{k+1}$ [caution: put the series in powers of $(x-a)$ first]

   (c) $\displaystyle\sum_{k=1}^{\infty} \frac{(x-1)^k}{k^3 3^k}$

   (d) $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k 3^{2k}(x-2)^k}{3k}$

   (e) $\displaystyle\sum_{k=0}^{\infty} k!(x-4)^k$

   (f) $\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k (x-1)^{4k}}{k!}$

   (g) $\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k (x-3)^{2k}}{9^k}$

   (h) $\displaystyle\sum_{k=1}^{\infty} \frac{k^k x^k}{k!}$

2. Find the radius of convergence and open interval of convergence of

$$\sum_{k=0}^{\infty} \frac{x^k}{e^k} .$$

   What function does this series represent?

3. Use the result in Example 12 to find $f^{(8)}(0)$ for $f(x) = \tan^2 x$. (Hint: use the identity $1 + \tan^2 x = \sec^2 x$.)

Answers

1. (a) $R = 10$; $\mathcal{I} = (-8, 12)$
   (b) $R = 1/3$; $\mathcal{I} = (1/3, 1)$
   (c) $R = 3$; $\mathcal{I} = (-2, 4)$
   (d) $R = 1/9$; $\mathcal{I} = (17/9, 19/9)$
   (e) $R = 0$; $\mathcal{I}$ does not exist (series converges only for $x = 4$)
   (f) $R = \infty$; $\mathcal{I} = (-\infty, \infty)$
   (g) $R = 3$; $\mathcal{I} = (0, 6)$
   (h) $R = 1/e$; $\mathcal{I} = (-1/e, 1/e)$
2. $R = e$; $\mathcal{I} = (-e, e)$; $f(x) = e/(e-x)$
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