Question 1:

(a) Use the trapezoind rule on 3 subintervals to approximate $\int_0^3 e^{x(1-x)(2-x)/6}\,dx$. (Simplify your final answer.)

$f(x) = e^{x(1-x)(2-x)/6}$.

$\Delta x = \frac{3-0}{3} = 1$

$\therefore \int_0^3 f(x)\,dx = \frac{\Delta x}{2}\left[f(0) + 2f(1) + 2f(2) + f(3)\right]$

$= \frac{1}{2}\left[e^0 + 2e^0 + 2e^0 + e\right]$

$= \boxed{\frac{5+e}{2}}$

[5]

(b) The second derivative of $f(x) = e^{x(1-x)(2-x)/6}$ can be shown to be between -1 and 15 on the interval $[0, 3]$. Use this information to determine an error bound $|E_{T_3}|$ on your approximation in part (a). Simplify your final answer.

(Recall: the error in using the trapezoid rule to approximate $\int_a^b f(x)\,dx$ using n subintervals is at most $\frac{K(b-a)^3}{12n^2}$ where $|f''(x)| \le K$ on $[a, b]$.)

$|f''(x)| \le 15$ on $[0,3]$, so $K = 15$.

$\therefore |E_{T_3}| \le \frac{15(3-0)^3}{12 \cdot 3^2} = \boxed{\frac{15}{4}}$

[5]

Question 2:

(a) Determine whether the following improper integral is convergent or divergent. If convergent, evaluate the integral. Make proper use of any required limits:

$$\int_{-1}^{1} \frac{e^x}{e^x - 1}\,dx$$ integrand discontinuous at $x=0$

$$= \int_{-1}^{0} \frac{e^x}{e^x-1}\,dx + \int_{0}^{1} \frac{e^x}{e^x-1}\,dx$$

$$\int_0^1 \frac{e^x}{e^x-1}\,dx = \lim_{a\to 0^+} \int_a^1 \frac{e^x}{e^x-1}\,dx$$

$$= \lim_{a\to 0^+} \Big[\ln|e^x-1|\Big]_a^1$$

$$= \lim_{a\to 0^+} \Big(\ln|e-1| - \underbrace{\ln|e^a-1|}_{\to -\infty}\Big)$$

$$= +\infty.$$

$\therefore \int_{-1}^{1} \frac{e^x}{e^x-1}\,dx$ diverges.

[5]

(b) Use the comparison theorem to determine whether the following integral is convergent or divergent:

$$\int_1^\infty \frac{1}{\sqrt{x}(1+x)}\,dx$$

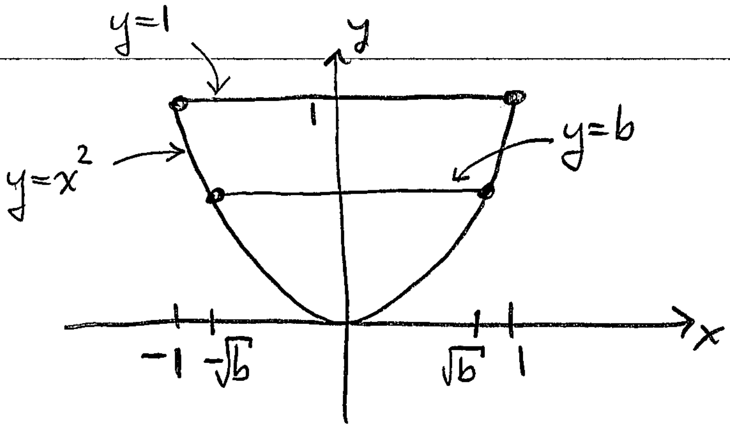
$$0 \leq \frac{1}{\sqrt{x}(1+x)} = \frac{1}{x^{1/2} + x^{3/2}} \leq \frac{1}{x^{3/2}} \quad \text{on } [1,\infty).$$

Since $\int_1^\infty \frac{1}{x^{3/2}}\,dx$ converges (p-integral, $p = \frac{3}{2} > 1$),

$\int_1^\infty \frac{1}{\sqrt{x}(1+x)}\,dx$ converges by the comparison theorem.

[5]

Question 3: The region bounded by the curves $y = x^2$ and $y = 1$ is divided into two regions by the line $y = b$. Find b if the two regions have equal areas.



Area between $y=1$ & $y=x^2$ is

$$\int_{-1}^{1} 1-x^2\,dx = \left[x-\frac{x^3}{3}\right]_{-1}^{1} = \frac{4}{3}$$

∴ need b so that

$$\int_{-\sqrt{b}}^{\sqrt{b}} b-x^2\,dx = \frac{1}{2}\left(\frac{4}{3}\right)$$

$$\therefore 2\int_0^{\sqrt{b}} b-x^2\,dx = \frac{2}{3}$$

$$2\left[bx-\frac{x^3}{3}\right]_0^{\sqrt{b}} = \frac{2}{3}$$

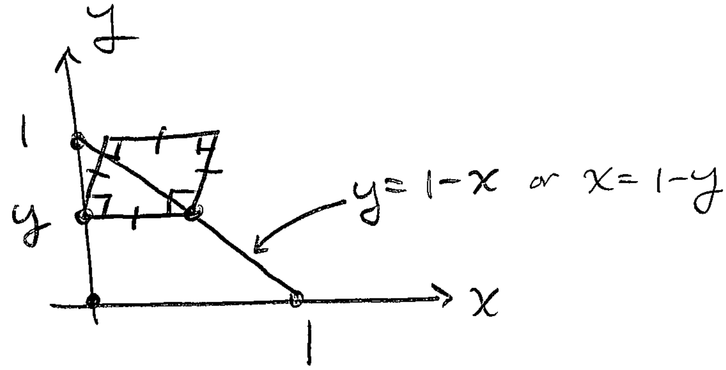
$$2\left(\frac{2}{3}b^{3/2}\right) = \frac{2}{3}$$

$$b^{3/2} = \left(\frac{2}{3}\right)\left(\frac{3}{4}\right) = \frac{1}{2}$$

$$\boxed{b = \left(\frac{1}{2}\right)^{2/3}}$$

[5]

Question 4: The base of the solid S is the triangular region in the xy-plane with vertices at $(0,0)$, $(1,0)$ and $(0,1)$. Cross-sections perpendicular to the y-axis are squares. Determine the volume of S.



$$\therefore A(y) = (1-y)^2$$

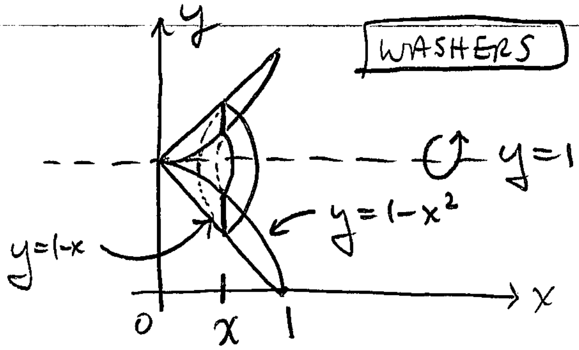
$$\therefore V = \int_{y=0}^{1} (1-y)^2\,dy$$

$$= \left[\frac{(1-y)^3}{-3}\right]_0^1 = \boxed{\frac{1}{3}}$$

[5]

Question 5: The region in the first quadrant bounded between $y = 1 - x^2$ and $y = 1 - x$ is rotated about the horizontal line $y = 1$. Determine the volume of the resulting solid (both the washer and cylindrical shells methods can be used here.)

WASHERS



$$A(x) = \pi\left[1-(1-x)\right]^2 - \pi\left[1-(1-x^2)\right]^2$$

$$= \pi\left(x^2 - x^4\right)$$

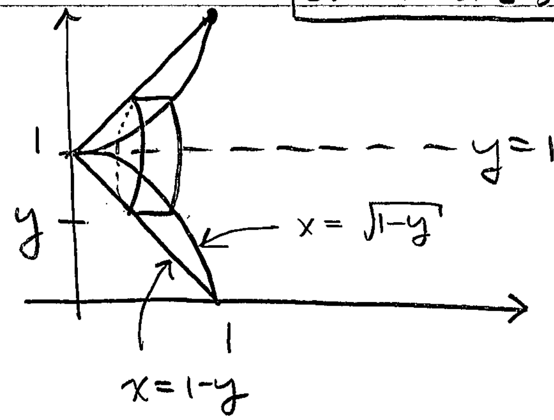
$$\therefore\ V = \pi\int_0^1 x^2 - x^4\, dx$$

$$= \pi\left[\frac{x^3}{3} - \frac{x^5}{5}\right]_0^1$$

$$= \boxed{\frac{2\pi}{15}}$$

OR

CYLINDRICAL SHELLS



$$\therefore\ V = \int_{y=0}^{1} 2\pi\,(1-y)\left[\sqrt{1-y} - (1-y)\right] dy$$

$$= 2\pi\int_0^1 (1-y)^{3/2} - (1-y)^2\, dy$$

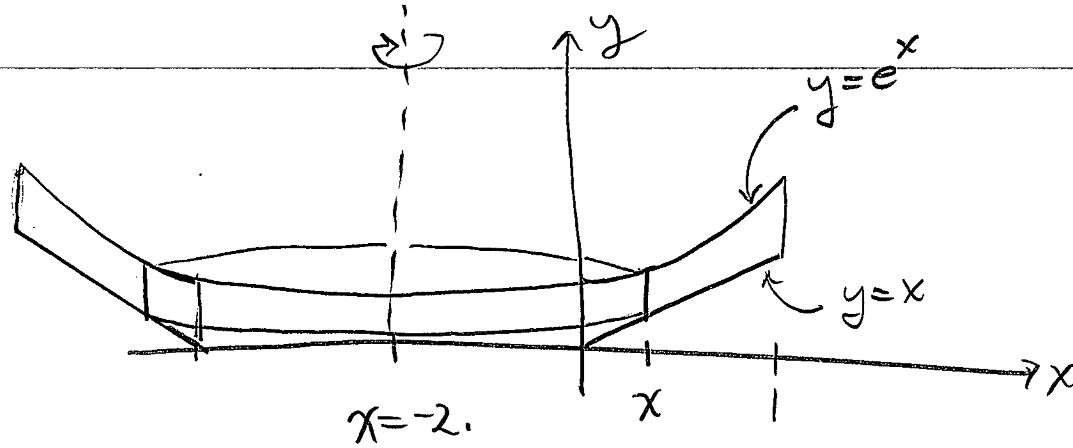
$$= 2\pi\left[-\frac{2}{5}(1-y)^{5/2} + \frac{(1-y)^3}{3}\right]_0^1$$

$$= 2\pi\left[\frac{2}{5}(1) - \frac{1}{3}\right]$$

$$= \boxed{\frac{2\pi}{15}}$$

[10]

Question 6: The region bounded by the curves $y = e^x$, $y = x$, $x = 0$ and $x = 1$ is rotated about the vertical line $x = -2$. Determine the volume of the resulting solid.



$$V = \int_0^1 2\pi (x+2)(e^x - x)\, dx$$

$$= 2\pi \int_0^1 \underbrace{xe^x}_{\text{by parts.}} + 2e^x - x^2 - 2x \, dx$$

$$= 2\pi \left[\overbrace{xe^x - e^x} + 2e^x - \frac{x^3}{3} - \frac{\cancel{2}x^2}{\cancel{2}} \right]_0^1$$

$$= 2\pi \left[\cancel{1 \cdot e} - \cancel{e} + 2e - \frac{1}{3} - 1 - 0 + 1 - 2 + 0 + 0 \right]$$

$$= \boxed{2\pi \left[2e - \frac{7}{3} \right]}$$

[10]