Question 1: Determine $\int_1^e x^3 \ln(x)\,dx = I$

$u = \ln(x) \quad dv = x^3 dx$

$du = \frac{1}{x} dx \quad v = \frac{x^4}{4}$

$\therefore I = \int_1^e u\,dv$

$= [uv]_1^e - \int_1^e v\,du$

$= \left[\frac{\ln(x)\,x^4}{4}\right]_1^e - \int_1^e \frac{x^4}{4} \cdot \frac{1}{x}\,dx$

$= \frac{\ln(e) \cdot e^4}{4} - \frac{\ln(1) \cdot 1^4}{4} - \frac{1}{4}\left[\frac{x^4}{4}\right]_1^e$

$= \frac{e^4}{4} - \frac{e^4}{16} + \frac{1}{16} = \boxed{\frac{3e^4+1}{16}}$

[5]

Question 2: Determine $\int (x^2 - 5x)\sin(x)\,dx = I$

$u = x^2 - 5x \quad dv = \sin(x)\,dx$

$du = (2x-5)\,dx \quad v = -\cos(x)$

$\therefore I = \int u\,dv$

$= uv - \int v\,du$

$$= (x^2-5x)(-\cos(x)) - \int \underbrace{-\cos(x)(2x-5)}_{\substack{u = 2x-5, \quad dv = \cos(x)\,dx \\ du = 2\,dx, \quad v = \sin(x)}} dx$$

$= -(x^2-5x)(\cos(x)) + uv - \int v\,du$

$= -(x^2-5x)\cos(x) + (2x-5)\sin(x) - \int 2\sin(x)\,dx$

$= \boxed{-(x^2-5x)\cos(x) + (2x-5)\sin(x) + 2\cos(x) + C}$

[5]

Question 3: Determine $\int \cos^5(x)\,dx$

$$= \int \cos^4(x)\ \cos(x)dx$$

$$= \int (1-\sin^2(x))^2 \cos(x)dx \quad \Big\}\ \text{let } u = \sin(x),\ du = \cos(x)$$

$$= \int (1-u^2)^2 du$$

$$= \int 1 - 2u^2 + u^4\ du$$

$$= u - \frac{2}{3}u^3 + \frac{u^5}{5} + C$$

$$= \boxed{\sin(x) - \frac{2}{3}\sin^3(x) + \frac{\sin^5(x)}{5} + C}$$

[5]

Question 4: Determine $\int \sec^4(3x)\tan^2(3x)\,dx$

$$= \int \sec^2(3x)\tan^2(3x)\sec^2(3x)\,dx$$

$$= \int (1+\tan^2(x))\tan^2(3x)\sec^2(3x)dx \quad \Big\}\ \text{let } u = \tan(3x),\ du = 3\sec^2(3x)dx$$

$$= \frac{1}{3}\int (1+u^2)u^2\,du$$

$$= \frac{1}{3}\left[\frac{u^3}{3} + \frac{u^5}{5}\right] + C$$

$$= \boxed{\frac{\tan^3(3x)}{9} + \frac{\tan^5(3x)}{15} + C}$$

[5]

Question 5: Determine $\int \frac{x^2}{\sqrt{9-x^2}}\,dx = I$

Let $x = 3\sin\theta$

$dx = 3\cos\theta\, d\theta$

$$\therefore I = \int \frac{9\sin^2\theta \quad 3\cos\theta}{\sqrt{9-9\sin^2\theta}}\, d\theta$$

$$= \int \frac{9\sin^2\theta\, \cancel{3\cos\theta}}{\cancel{3\cos\theta}}\, d\theta$$

$$= \frac{9}{2}\int 1-\cos(2\theta)\, d\theta$$

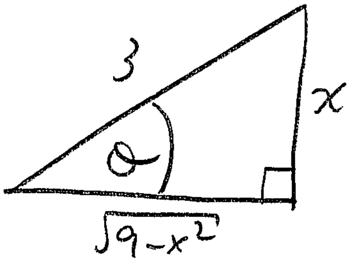
$$= \frac{9}{2}\left[\theta - \frac{\sin(2\theta)}{2}\right] + C$$

$$= \frac{9}{2}\left[\theta - \frac{\cancel{2}\sin\theta\cos\theta}{\cancel{2}}\right] + C$$

$$= \frac{9}{2}\left[\arcsin\left(\frac{x}{3}\right) - \frac{x}{3}\cdot\frac{\sqrt{9-x^2}}{3}\right] + C$$

$$= \boxed{\frac{9}{2}\arcsin\left(\frac{x}{3}\right) - \frac{x\sqrt{9-x^2}}{2} + C}$$

$\sin\theta = \frac{x}{3}$



[10]

Question 6: Determine $\int \frac{x^2}{(x-1)(x+1)^2}\,dx = I$

$$\frac{x^2}{(x-1)(x+1)^2} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$$

$$= \frac{A(x+1)^2 + B(x-1)(x+1) + C(x-1)}{(x-1)(x+1)^2}$$

$$= \frac{(A+B)x^2 + (2A+C)x + (A-B-C)}{(x-1)(x+1)^2}$$

$\therefore$ $A+B=1$ ① ① $\Rightarrow B = 1-A$

$2A+C=0$ ② ② $\Rightarrow C = -2A$

$A-B-C=0$ ③ ③ $\Rightarrow A-(1-A)-(-2A)=0$

$4A = 1$

$A = \frac{1}{4}$

$\therefore B = 1-\frac{1}{4} = \frac{3}{4}$

$\therefore C = -2A = -2\left(\frac{1}{4}\right) = \frac{-1}{2}$

$$\therefore I = \int \frac{(1/4)}{x-1} + \frac{(3/4)}{x+1} + \frac{(-1/2)}{(x+1)^2}\,dx$$

$$= \boxed{\frac{1}{4}\ln|x-1| + \frac{3}{4}\ln|x+1| + \frac{1}{2}\,\frac{1}{x+1} + C}$$

[10]

Question 7: Determine $\int \frac{x^3+x+2}{x^2+4}\,dx = I$

$$\begin{array}{r} x \\ x^2+0x+4 \overline{\smash{)}\, x^3+0x^2+x+2} \\ -(x^3+0x^2+4x) \\ \hline -3x+2 \end{array}$$

$$\therefore I = \int x + \frac{-3x+2}{x^2+4}\,dx$$

$$= \int x\,dx - \frac{3}{\textcircled{2}}\underbrace{\int \frac{\textcircled{2}x}{x^2+4}\,dx}_{u=x^2+4;\ du=2x\,dx} + 2\underbrace{\int \frac{1}{x^2+2^2}\,dx}_{\text{Formula \#18}}$$

$$= \boxed{\frac{x^2}{2} - \frac{3}{2}\ln|x^2+4| + \arctan\left(\frac{x}{2}\right) + C}$$

[5]

Question 8: Determine $\int \frac{1}{x^2+4x+8}\,dx$

$$= \int \frac{1}{(x+2)^2+2^2}\,dx \quad \Big\} \text{ Formula 18.}$$

$$= \boxed{\frac{1}{2}\arctan\left(\frac{x+2}{2}\right) + C}$$

[5]