Question 1:

(a) Use a linear approximation $T_1(x)$ for $f(x) = \dfrac{1}{\sqrt{1+x}}$ to approximate $f(1/10)$. Express your answer as a single simplified fraction.

$$f(x) = (1+x)^{-\frac{1}{2}}; \quad f(0) = 1$$

$$f'(x) = -\tfrac{1}{2}(1+x)^{-3/2}; \quad f'(0) = -\tfrac{1}{2}$$

$$\therefore T_1(x) = f(0) + f'(0)\,x = 1 - \tfrac{1}{2}x$$

$$\therefore f(\tfrac{1}{10}) \approx T_1(\tfrac{1}{10}) = 1 - (\tfrac{1}{2})(\tfrac{1}{10})$$

$$= 1 - \frac{1}{20}$$

$$= \frac{20-1}{20}$$

$$= \boxed{\frac{19}{20}}$$

[5]

(b) Give an error bound for your approximation in part (a). Again, express your answer as a single simplified fraction.

$$f''(z) = \frac{3}{4}(1+z)^{-\frac{5}{2}} = \frac{3}{4(1+z)^{5/2}}$$

$$R_1(\tfrac{1}{10}) = \frac{f''(z)(\frac{1}{10})^2}{2!} \quad \text{where} \quad 0 < z < \tfrac{1}{10}$$

$$\therefore |R_1(\tfrac{1}{10})| = \left|\frac{3}{4(1+z)^{5/2}}\right| \cdot \left|\frac{1}{2}\right| \cdot \left(\frac{1}{10}\right)^2, \quad 0 < z < \tfrac{1}{10}$$

$$\leq (\tfrac{3}{4})(\tfrac{1}{2})(\tfrac{1}{100})$$

$$= \boxed{\frac{3}{800}}$$

[5]

Question 2:

(a) Find the Taylor polynomial of degree 2 for $f(x) = x^2 \ln(x)$ at $a = 1$.

$$f(x) = x^2 \ln(x) \quad ; \quad f(1) = 0$$

$$f'(x) = 2x \ln(x) + x^2\left(\frac{1}{x}\right)$$

$$= 2x \ln(x) + x \quad ; \quad f'(1) = 1$$

$$f''(x) = 2\ln(x) + 2x\left(\frac{1}{x}\right) + 1$$

$$= 2\ln(x) + 3 \quad ; \quad f''(1) = 3$$

$$\therefore T_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2$$

$$= 0 + (1)(x-1) + \frac{3}{2}(x-1)^2$$

$$= \boxed{(x-1) + \frac{3}{2}(x-1)^2}$$

[5]

(b) Suppose $T_2(x)$ in part (a) is used to approximate $f(4/5)$. Give an error bound on the approximation. Express your answer as a single simplified fraction. (Note: you are not being asked to find the approximation to $f(4/5)$ here, but only the error bound associated with the approximation.)

$$f'''(z) = \frac{2}{z}$$

$$\left|R_2\left(\frac{4}{5}\right)\right| = \left|\frac{f'''(z)\left(\frac{4}{5}-1\right)^3}{3!}\right|, \quad \frac{4}{5} < z < 1$$

$$= \left|\left(\frac{2}{z}\right)\left(\frac{1}{6}\right)\left(\frac{-1}{5}\right)^3\right|, \quad \frac{4}{5} < z < 1$$

$$\leq \frac{1\cancel{2}}{\left(\frac{4}{\cancel{5}}\right)} \cdot \frac{1}{\cancel{6}3} \cdot \frac{1}{5^{\cancel{3}2}}$$

$$= \boxed{\frac{1}{300}}$$

[5]

Question 3:

(a) Find the first four nonzero terms of the Maclaurin series for $f(x) = x^2 \cos(2x)$. You may leave the terms of your answer in factored form.

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$\therefore \cos(2x) = 1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \frac{(2x)^6}{6!} + \cdots$$

$$\therefore x^2\cos(2x) = \boxed{x^2 - \frac{2^2 x^4}{2!} + \frac{2^4 x^6}{4!} - \frac{2^6 x^8}{6!} + \cdots}$$

[5]

(b) Again for $f(x) = x^2 \cos(2x)$, determine $f^{(12)}(0)$. You may leave your answer as a fraction in factored form.

Using (a), the degree 12 term of the Maclaurin series for $x^2\cos(2x)$ is $-\frac{2^{10}}{10!}x^{12}$.

$$\therefore \frac{f^{(12)}(0)}{12!}x^{12} = -\frac{2^{10}}{10!}x^{12}$$

$$\therefore f^{(12)}(0) = \boxed{-2^{10}\frac{12!}{10!}} \quad \underline{\text{or}} \quad \boxed{-2^{10}(12)(11)}.$$

[5]

Question 4:

(a) Find the first four nonzero terms of the Maclaurin series for $f(x) = \frac{x^3}{1+x}$.

& state I.

$$\frac{1}{1-x} = 1+x+x^2+x^3+\cdots$$

$$\therefore \frac{1}{1+x} = 1-x+x^2-x^3+\cdots$$

$$\boxed{\therefore \frac{x^3}{1+x} = x^3-x^4+x^5-x^6+\cdots \quad , \quad I=(-1,1)}$$

[3]

(b) Find the Maclaurin polynomial of degree 17 for $g(x) = x^2e^{-x^5}$

$$e^x = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots$$

$$\therefore e^{-x^5} = 1-x^5+\frac{x^{10}}{2!}-\frac{x^{15}}{3!}+\cdots$$

$$\therefore x^2e^{-x^5} = x^2-x^7+\frac{x^{12}}{2!}-\frac{x^{17}}{3!}+\cdots$$

$$\therefore \boxed{T_{17}(x) = x^2-x^7+\frac{x^{12}}{2!}-\frac{x^{17}}{3!}}$$

[3]

& state I.

(c) Find the the first four nonzero terms of the Taylor series for $h(x) = \frac{1}{3-2x}$ about $a = 1$. (Hint: think of the series for $\frac{1}{1-x}$.)

$$h(x) = \frac{1}{3-2x}$$

$$= \frac{1}{3-2(x-1)-2}$$

$$= \frac{1}{1-2(x-1)}$$

$$= \boxed{1+2(x-1)+2^2(x-1)^2+2^3(x-1)^3+\cdots}$$

$$|2(x-1)|<1$$

$$|x-1| < \frac{1}{2}$$

$$\therefore I = \left(\frac{1}{2}, \frac{3}{2}\right),$$

[4]

Question 5: Find the first three nonzero terms of the Maclaurin series for $f(x) = \arctan(x) \cdot e^x$

$$f(x) = \left[x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots\right]\left[1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots\right]$$

$$= x + x^2 + \left(\frac{1}{2} - \frac{1}{3}\right)x^3 + \left(\frac{1}{3!} + \frac{-1}{3}\right)x^4 + \cdots$$

$$= \boxed{x + x^2 + \frac{1}{6}x^3 - \frac{1}{6}x^4 + \cdots}$$

[5]

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Question 6: Evaluate the limit $\lim_{x \to 0} \dfrac{\sin(x^2) - x^2\cos(x^2)}{e^{(x^6)} - 1}$.

$$= \lim_{x\to 0} \frac{\left[x^2 - \frac{(x^2)^3}{3!} + \frac{(x^2)^5}{5!}\right] - x^2\left[1 - \frac{(x^2)^2}{2!} + \frac{(x^2)^4}{4!} - \cdots\right]}{\left[\cancel{1} + (x^6) + \frac{(x^6)^2}{2!} + \cdots\right] - \cancel{1}}$$

$$= \lim_{x\to 0} \frac{\left[\cancel{x^2} - \frac{x^6}{3!} + \frac{x^{10}}{5!}\right] - \left[\cancel{x^2} - \frac{x^6}{2!} + \frac{x^{10}}{4!} + \cdots\right]}{x^6 + \frac{x^{12}}{2!} + \frac{x^{18}}{3!} + \cdots}$$

$$= \lim_{x\to 0} \frac{\cancel{x^6}\left[-\frac{1}{3!} + \frac{x^4}{5!} - \cdots\right] - \cancel{x^6}\left[\frac{-1}{2!} + \frac{x^4}{4!} + \cdots\right]}{\cancel{x^6}\left[1 + \frac{x^6}{2!} + \frac{x^{12}}{3!} - \cdots\right]}$$

$$= -\frac{1}{3!} + \frac{1}{2!} = \boxed{\frac{1}{3}}$$

[5]