Question 1:

(a) Given that $y_1(x) = e^x$ is one solution of

$$xy'' - (x+1)y' + y = 0, \quad x > 0,$$

use reduction of order to find a second linearly independent solution $y_2(x)$. (Do not use "the formula", but rather, find $y_2(x)$ from first principles. You may check your result using the formula however.)

$y_2 = uy_1 = ue^x \Rightarrow y_2' = u'e^x + ue^x$, $y_2'' = u''e^x + 2u'e^x + ue^x$

Inserting into DE:

$x[u''e^x + 2u'e^x + ue^x] - (x+1)[u'e^x + ue^x] + ue^x = 0$

$xe^x u'' + 2xe^x u' + \cancel{xe^x u} - xe^x u' - \cancel{xe^x u} - e^x u' - \cancel{e^x u} + \cancel{e^x u} = 0$

$\therefore\ e^x[xu'' + (x-1)u'] = 0 \ \}$ let $w = u'$, $w' = u''$

$\Rightarrow \quad xw' + (x-1)w = 0$

$\Rightarrow \int \frac{1}{w}\,dw = \int \frac{1-x}{x}\,dx$

$\Rightarrow \ln|w| = \ln(x) - x + C_1 \}$ note $x > 0$

$\Rightarrow \quad w = C_2 x e^{-x}$

$\Rightarrow \quad u = \int C_2 x e^{-x}\,dx = C_2 \int x\, d[-e^{-x}] = -C_2 x e^{-x} + C_2 \int e^{-x}\,dx$

$= -C_2 x e^{-x} - C_2 e^{-x} + C_3$

$\therefore\ y_2 = uy_1 = -C_2 x - C_2 + C_3 e^x = \boxed{x+1}$ taking $C_2 = -1$, $C_3 = 0$

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(b) Show that $y_1(x)$ and $y_2(x)$ from part (a) are linearly independent.

$W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^x & x+1 \\ e^x & 1 \end{vmatrix} = \cancel{e^x} - xe^x - \cancel{e^x} = \underset{\neq 0}{-xe^x}$ on $(0, \infty)$.

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Question 2:

(a) Find the general solution of $y'' - 2y' + y = 0$

$$r^2 - 2r + 1 = 0$$

$$(r-1)^2 = 0$$

$$r = 1, 1$$

$$\therefore \boxed{y = c_1 e^x + c_2 x e^x.}$$

[3]

(b) Suppose you are finding the general solution of $y'' - 2y' + y = x^2 e^x$ using the method of undetermined coefficients. Making use of your result in part (a), state the form of the trial solution y_p. (Note: do not determine the constants in y_p, simply state the form you would use.)

$$\boxed{y_p = x^2 (Ax^2 + Bx + C) e^x}$$

required since, otherwise, a constant multiple of a term of y_p would appear in y_c.

[3]

Question 3: Calculate $\mathcal{L}\{f(t)\}$ where $f(t) = te^{2t}$ (do not use tables).

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} t e^{2t} dt$$

$$= \int_0^\infty t \, d\left[\frac{e^{-(s-2)t}}{-(s-2)}\right]$$

$$= \cancel{\left[\frac{t e^{-(s-2)t}}{-(s-2)}\right]_0^\infty}^{\,0} + \frac{1}{(s-2)} \int_0^\infty e^{-(s-2)t} dt$$

$$= \frac{-1}{(s-2)^2} \left[e^{-(s-2)t}\right]_0^\infty$$

$$= \boxed{\frac{1}{(s-2)^2}}$$

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Question 4: Solve:

$$x^2y'' + xy' - y = \ln(x)$$

- For $x^2y''+xy'-y=0$: Let $y=x^m$, $y'=mx^{m-1}$, $y''=m(m-1)x^{m-2}$

$\therefore x^2[m(m-1)x^{m-2}]+x[mx^{m-1}]-x^m=0$

$x^m[m^2-1]=0$

$m^2=1 \Rightarrow m=1,-1$

$\therefore y_c = c_1 x + c_2 x^{-1}$

- For y_p: variation of parameters with $y_1=x$, $y_2=\frac{1}{x}$, $f(x)=\frac{\ln(x)}{x^2}$

$$W=\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} x & \frac{1}{x} \\ 1 & \frac{-1}{x^2} \end{vmatrix} = \frac{-2}{x}$$

$$u_1' = \frac{-y_2 f(x)}{W} = \frac{(-\frac{1}{x})\frac{\ln(x)}{x^2}}{-\frac{2}{x}} = \frac{\ln(x)}{2x^2}$$

$$\therefore u_1 = \int \ln(x)\, d\left[-\frac{1}{2x}\right] = -\frac{\ln(x)}{2x} + \int \frac{1}{2x}\cdot\frac{1}{x}\,dx$$

$$= \frac{-\ln(x)}{2x} - \frac{1}{2x}$$

$$u_2' = \frac{y_1 f(x)}{W} = \frac{\left(x\frac{\ln(x)}{x^2}\right)}{-\frac{2}{x}} = \frac{-1}{2}\ln(x)$$

$$\therefore u_2 = \int -\frac{1}{2}\ln(x)\,dx = -\frac{1}{2}x\ln(x) + \frac{1}{2}x$$

$$\therefore y_p = u_1y_1+u_2y_2 = \left(-\frac{\ln(x)}{2x} - \frac{1}{2x}\right)\cdot x + \left(-\frac{1}{2}x\ln(x)+\frac{1}{2}x\right)\left(\frac{1}{x}\right)$$

$$= \frac{-\ln(x)}{2} \cancel{-\frac{1}{2}} - \frac{\ln(x)}{2} \cancel{+\frac{1}{2}}$$

$$= -\ln(x)$$

$$\therefore y = y_c + y_p = \boxed{c_1 x + c_2 x^{-1} - \ln(x)}$$

[10]

Question 5: A mass of 1 slug, when attached to a spring, stretches it 2 feet then comes to rest at equilibrium. Starting at $t = 0$ and external force of $f(t) = 8\sin(4t)$ is applied to the system. The surrounding medium imparts a damping force equal to 8 times the instantaneous velocity. Find the equation of motion. (Use any method you like to solve the resulting differential equation.)

$F = (1)(32) = 32$ lbs.

$-32 = -k(2) \Rightarrow k = 16.$

$\beta = 8$

$\therefore\ 1x'' + 8x' + 16x = 8\sin(4t), \quad x(0) = 0, \quad x'(0) = 0$

- For x_c: $r^2 + 8r + 16 = 0 \Rightarrow (r+4)^2 = 0 \Rightarrow r = -4, -4$

  $\therefore\ x_c = c_1 e^{-4t} + c_2 t e^{-4t}$

- For x_p: undetermined coefficients: $x_p = A\cos(4t) + B\sin(4t)$

  $x_p' = -4A\sin(4t) + 4B\cos(4t)$

  $x_p'' = -16A\cos(4t) - 16B\sin(4t)$

  Inserting into DE:

  $$-16A\cos(4t) - 16B\sin(4t) + 8[-4A\sin(4t) + 4B\cos(4t)] + 16[A\cos(4t) + B\sin(4t)] = 8\sin(4t)$$

  $$\therefore\ \left.\begin{array}{ll} \cos(4t): & 32B = 0 \\ \sin(4t): & -32A = 8 \end{array}\right\} \therefore\ A = -\frac{1}{4},\ B = 0$$

$\therefore\ x = x_c + x_p = c_1 e^{-4t} + c_2 t e^{-4t} - \frac{1}{4}\cos(4t).$

$x' = -4c_1 e^{-4t} + c_2 e^{-4t} - 4c_2 t e^{-4t} + \sin(4t).$

$x(0) = 0 \Rightarrow c_1 - \frac{1}{4} = 0 \Rightarrow c_1 = \frac{1}{4}$

$x'(0) = 0 \Rightarrow -4c_1 + c_2 = 0 \Rightarrow c_2 = 4c_1 = 1$

$$\boxed{\therefore\ x(t) = \frac{1}{4}e^{-4t} + te^{-4t} - \frac{1}{4}\cos(4t)}$$

[10]

Question 6: Solve using the Laplace transform:

$$y'' + 4y = 2e^{-t}, \quad y(0) = 1, \quad y'(0) = 0$$

$$\mathcal{L}\{y''+4y\} = \mathcal{L}\{2e^{-t}\}$$

$$s^2Y - s\cancel{y(0)}^{1} - \cancel{y'(0)}^{0} + 4Y = 2\left(\frac{1}{s+1}\right)$$

$$Y(s^2+4) = s + 2\frac{1}{s+1}$$

$$Y = \underbrace{\frac{s}{s^2+4}}_{\text{Formula \#8}} + 2\underbrace{\frac{1}{(s+1)(s^2+4)}}_{\text{partial fractions.}}$$

$$\frac{1}{(s+1)(s^2+4)} = \frac{A}{s+1} + \frac{Bs+C}{s^2+4} = \frac{(A+B)s^2 + (B+C)s + (4A+C)}{(s+1)(s^2+4)}$$

$$\therefore A+B = 0 \Rightarrow B = -A$$

$$B+C = 0 \Rightarrow C = -B = A$$

$$4A+C = 1 \Rightarrow 4A+A = 1 \Rightarrow A = \frac{1}{5}, \quad B = \frac{-1}{5}, \quad C = \frac{1}{5}$$

$$\therefore Y = \frac{s}{s^2+4} + \frac{2}{5}\frac{1}{s+1} - \frac{2}{5}\frac{s}{s^2+4} + \frac{2}{5}\frac{1}{s^2+4}$$

$$Y = \frac{3}{5}\frac{s}{s^2+2^2} + \frac{1}{5}\frac{2}{s^2+2^2} + \frac{2}{5}\frac{1}{s+1}$$

$$\boxed{\therefore\ y = \frac{3}{5}\cos(2t) + \frac{1}{5}\sin(2t) + \frac{2}{5}e^{-t}}$$

[10]