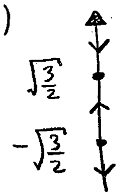
Question 1 [10]: For this question use the differential equation $\frac{dv}{dt} = 3 - 2v^2$

(a) Determine (i) the equilibrium solutions, (ii) sketch the one dimensional phase portrait, and (iii) classify each equilibrium solution as asymptotically stable, unstable or semi-stable.

(i) $3 - 2v^2 = 0 \Rightarrow v^2 = \frac{3}{2} \Rightarrow v = \pm\sqrt{\frac{3}{2}}$,

(ii)



(iii) $v = \sqrt{\frac{3}{2}}$ is asymptotically stable.

$v = -\sqrt{\frac{3}{2}}$ is unstable.

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(b) If $v(0) = -1$, what is $\lim_{t\to\infty} v(t)$?

$\lim_{t\to\infty} v(t) = \boxed{\sqrt{\frac{3}{2}}.}$

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(c) If $v(0) = 2$, use Euler's method with $h = 0.05$ to approximate $v(0.15)$ to three decimals.

$f(t, v) = 3 - 2v^2$

$\underline{h = 0.05}$: $v_{n+1} = v_n + h\, f(t_n, v_n) = v_n + (0.05)(3 - 2v_n^2)$.

n	t_n	v_n	v_{n+1}
0	0	2	1.75
1	0.05	1.75	1.59375
2	0.10	1.59375	$\approx$ 1.490

$\boxed{\therefore\ v(0.15) \approx 1.490}$

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Question 2: Solve the IVP $\dfrac{dy}{dx} = 8x^3e^{-2y}$, $y(1) = 0$. State the solution in explicit form and give the largest interval on which the solution is valid.

$$\int e^{2y}dy = \int 8x^3 dx$$

$$\frac{1}{2}e^{2y} = 2x^4 + C_1$$

$$e^{2y} = 4x^4 + C_2$$

$\underline{y(1)=0}$: $e^0 = 4(1)^4 + C_2 \Rightarrow C_2 = -3.$

$$\therefore e^{2y} = 4x^4 - 3$$

$$\boxed{y = \frac{1}{2}\ln\left(4x^4-3\right)}$$

For I: $4x^4 - 3 > 0$

$x^4 > \frac{3}{4}$

$\left.\begin{matrix} x > \left(\frac{3}{4}\right)^{1/4} \end{matrix}\right\}$ must be positive since $x_0 = 1 > 0$

$\underline{\text{or}}$ $x < -\left(\frac{3}{4}\right)^{1/4}$

$$\boxed{\therefore I = \left(\left(\tfrac{3}{4}\right)^{1/4}, \infty\right).}$$

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Question 3: Solve the IVP $t^2\dfrac{dx}{dt} + 3tx = \dfrac{\ln(t)}{t^2} + 1$, $x(1) = 0$. State the solution in explicit form and give the largest interval on which the solution is valid.

$$\left.\frac{dx}{dt} + \left(\frac{3}{t}\right)x = \frac{\ln(t)}{t^4} + \frac{1}{t^2}\right\} \quad P(t) = \frac{3}{t}, \; f(t) = \frac{\ln(t)}{t^4} + \frac{1}{t^2}$$

$$\boxed{\therefore \; I = (0, \infty)}$$

$$\mu(t) = \exp\left(\int \frac{3}{t}\,dt\right) = \exp\left(3\ln|t|\right) = |t|^3 = t^3 \text{ since}.$$

$$\therefore \frac{d}{dt}\left[t^3x\right] = \frac{\ln(t)}{t} + t$$

$$t^3x = \int \frac{\ln(t)}{t} + t \; dt = \frac{[\ln(t)]^2}{2} + \frac{t^2}{2} + C.$$

$\underline{x(1)=0}$: $0 = \cancel{\dfrac{[\ln(1)]^2}{2}}_{0} + \dfrac{1^2}{2} + C \Rightarrow C = -\dfrac{1}{2}$

$$\boxed{\therefore \; x = \frac{[\ln(t)]^2}{2t^3} + \frac{1}{2t} - \frac{1}{2t^3}}$$

[5]

Question 4: Solve: $\underbrace{(ye^{xy} - \frac{1}{y})}_{M(x,y)}\, dx + \underbrace{(xe^{xy} + \frac{x}{y^2})}_{N(x,y)}\, dy = 0$. You may leave your solution in implicit form.

$$\frac{\partial M}{\partial y} = e^{xy} + xye^{xy} + \frac{1}{y^2} = \frac{\partial N}{\partial x} \quad \text{, so exact}$$

$$f(x,y) = \int \left(ye^{xy} - \frac{1}{y}\right) dx = e^{xy} - \frac{x}{y} + g(y)$$

$$\frac{\partial f}{\partial y} = xe^{xy} + \frac{x}{y^2} + g'(y) = N(x,y) \Rightarrow g'(y) = 0 \Rightarrow g(y) = C_1$$

$$\therefore \boxed{e^{xy} - \frac{x}{y} = C}$$

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Question 5: Solve: $\frac{dy}{dx} - 5y = -\frac{5}{2}xy^3$, and also state constant solutions, if any. You may leave your solution in implicit form.

Constant solution : $\boxed{y=0}$

Equation is Bernoulli, $n=3$.

Let $u = y^{1-3} = y^{-2} \Rightarrow y = u^{-1/2} \Rightarrow \frac{dy}{dx} = -\frac{1}{2}u^{-3/2}\frac{du}{dx}$.

Equation becomes $-\frac{1}{2}u^{-3/2}\frac{du}{dx} - 5u^{-1/2} = \frac{-5}{2}xu^{-3/2}$

$$\Rightarrow \frac{du}{dx} + 10u = 5x$$

$$\mu(x) = \exp\left(\int 10\, dx\right) = e^{10x}$$

$$\therefore \frac{d}{dx}\left[e^{10x}u\right] = 5xe^{10x}$$

$$\therefore e^{10x}u = \int 5xe^{10x}dx = \frac{1}{2}\int x\, d\left[e^{10x}\right] \quad \Big\} \text{ by parts: } u = x,\ dv = e^{10x}dx$$

$$= \frac{1}{2}\left[xe^{10x} - \int e^{10x}dx\right]$$

$$= \frac{1}{2}xe^{10x} - \frac{1}{20}e^{10x} + C$$

$$\therefore u = \frac{1}{2}x - \frac{1}{20} + Ce^{-10x}$$

$$\therefore \boxed{y^{-2} = \frac{1}{2}x - \frac{1}{20} + Ce^{-10x}}$$

[5]

Question 6: Solve the IVP $\frac{dy}{dx} = 1 + e^{y-x+5}$, $y(5) = 0$. State your solution in explicit form.

let $u = y - x + 5$

$\frac{du}{dx} = \frac{dy}{dx} - 1$

$\therefore \frac{du}{dx} + \cancel{1} = \cancel{1} + e^{u}$

$\int e^{-u} du = \int dx$

$-e^{-u} = x + C$

$-e^{x-y-5} = x + C$

$y(5) = 0 \Rightarrow -e^{5-0-5} = 5 + C$

$\Rightarrow -1 = 5 + C$

$\Rightarrow C = -6$

$\therefore -e^{x-y-5} = x - 6$

$e^{x-y-5} = 6 - x$

$x - y - 5 = \ln(6-x)$

$\boxed{y = x - 5 - \ln(6-x)}$

[5]

Question 7: Determine the most general form of the function $M(x, y)$ if the following equation is exact:

$$M(x, y)\, dx + \underbrace{(\sin(x)\cos(y) - xy - e^{-y})}_{N}\, dy = 0$$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = \cos(x)\cos(y) - y$

$\therefore M = \int \cos(x)\cos(y) - y \, dy = \boxed{\cos(x)\sin(y) - \frac{y^2}{2} + g(x)}$

[5]

Question 8 [10]: The Gompertz differential equation

$$\frac{dP}{dt} = P(a - b\ln P)$$

(with a and b constant) is a modification of the logistic differential equation used for population growth.

(a) Find an explicit solution to this differential equation if a and b are both positive and $P(0) = 1$.

$$\int \frac{1}{P(a-b\ln P)}\,dP = \int dt \quad \Big\} \; u = a - b\ln(P). \quad du = \frac{-b}{P}\,dP$$

$$-\frac{1}{b}\ln\left(a - b\ln P\right) = t + C_1 \quad \Big\} (*)$$

$P(0)=1$: $\quad -\frac{1}{b}\ln\left(a - b\,\underbrace{\cancel{\ln(1)}}_{0}\right) = 0 + C_1$

$$C_1 = \frac{-\ln(a)}{b}$$

$$\therefore -\frac{1}{b}\ln(a - b\ln P) = t - \frac{\ln(a)}{b}$$

$$\Rightarrow t = \frac{\ln(a) - \ln(a - b\ln P)}{b}$$

$$= \frac{1}{b}\ln\left[\frac{a}{a - b\ln P}\right]$$

$$= \ln\left[\frac{a}{a - b\ln P}\right]^{1/b}$$

$$\therefore e^{t} = \left[\frac{a}{a - b\ln P}\right]^{\frac{1}{b}}$$

$$\therefore e^{bt} = \frac{a}{a - b\ln P}$$

$$\Rightarrow a - b\ln P = ae^{-bt}$$

$$\Rightarrow \frac{a(1 - e^{-bt})}{b} = \ln P$$

$$\Rightarrow \boxed{P(t) = e^{\frac{a}{b}(1 - e^{-bt})}}$$

(*) Note:

$a - b\ln(P) > 0 \iff P < e^{a/b}$, which is the case since $P(0) = 1$.

$\therefore$ no need for absolute value.

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(b) Will the population ever exceed its initial value of $P(0) = 1$? Explain.

$\boxed{\text{Yes.}}$ From $\frac{dP}{dt} = P(a - b\ln P)$, $\frac{dP}{dt} > 0$ at $t = 0$, so P increases from 1 as t increases from 0.

Alternatively, using (a), $P'(t) > 0$ for all t, so again P increases from 1.

[2]

(c) What is $\lim_{t\to\infty} P(t)$?

$$\lim_{t\to\infty} P(t) = \lim_{t\to\infty} e^{\frac{a}{b}(1 - e^{-bt})} = e^{\frac{a}{b}(1-0)} = \boxed{e^{a/b}}$$

[2]