Question 1:

(a) Given that $y_1(x) = 1/x$ is one solution of

$$2x^2y'' + 3xy' - y = 0, \qquad x > 0,$$

use reduction of order to find a second linearly independent solution $y_2(x)$. (Do not use "the formula", but rather, find $y_2(x)$ from first principles. You may check your result using the formula however.)

$y_2 = uy_1 = u(\frac{1}{x}) \Rightarrow y_2' = u'(\frac{1}{x}) + u(-\frac{1}{x^2})$, $y_2'' = u''(\frac{1}{x}) + 2u'(-\frac{1}{x^2}) + 2u(\frac{1}{x^3})$

Inserting into DE:

$$2x^2\left[u''(\tfrac{1}{x}) + 2u'(-\tfrac{1}{x^2}) + \cancel{2u(\tfrac{1}{x^3})}\right] + 3x\left[u'(\tfrac{1}{x}) + \cancel{u(-\tfrac{1}{x^2})}\right] - \cancel{u(\tfrac{1}{x})} = 0$$

$$\left.\begin{array}{l} 2xu'' - u' = 0 \\ 2xw' - w = 0 \end{array}\right\} \text{let } w = u',\ w' = u''$$

$$\Rightarrow \frac{w'}{w} = \frac{1}{2x}$$

$$\Rightarrow \int \frac{1}{w}\,dw = \frac{1}{2}\int \frac{1}{x}\,dx$$

$$\ln|w| = \frac{1}{2}\ln|x| + C_1$$

$$|w| = C_2|x|^{1/2} = C_2x^{1/2} \text{ since } x > 0$$

$$\therefore\ w = C_3x^{1/2}$$

$$\therefore\ u = \int C_3x^{1/2}\,dx = C_4x^{3/2} + C_5$$

$$\therefore\ y_2 = uy_1 = (C_4x^{3/2} + C_5)(x^{-1}) = \boxed{x^{1/2}} \text{ taking } C_4 = 1,\ C_5 = 0$$

[6]

(b) Show that $y_1(x)$ and $y_2(x)$ from part (a) are linearly independent.

$$W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \frac{1}{x} & x^{1/2} \\ -\frac{1}{x^2} & \frac{1}{2x^{1/2}} \end{vmatrix} = \frac{1}{2x^{3/2}} + \frac{1}{x^{3/2}} = \frac{3}{2}\frac{1}{x^{3/2}}$$

$$\neq 0 \text{ on } (0,\infty).$$

[4]

Question 2: Find the general solution for each of the following:

(a) $y'' + y' + y = 0$

$r^2 + r + 1 = 0$

$r = \dfrac{-1 \pm \sqrt{1-4(1)(1)}}{2} = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$

$\boxed{\therefore\ y = c_1 e^{-\frac{1}{2}t}\cos\left(\frac{\sqrt{3}}{2}t\right) + c_2 e^{-\frac{1}{2}t}\sin\left(\frac{\sqrt{3}}{2}t\right).}$

[3]

(b) $y'' + 2y' + y = 0$

$r^2 + 2r + 1 = 0$

$(r+1)^2 = 0$

$r = -1, -1$

$\boxed{\therefore\ y = c_1 e^{-t} + c_2 t e^{-t}}$

[3]

(c) $2y'' + \dfrac{5}{x}y' + \dfrac{y}{x^2} = 0$

$\Rightarrow\ 2x^2y'' + 5xy' + y = 0$

letting $y = x^m \Rightarrow y' = mx^{m-1}, y'' = m(m-1)x^{m-2}$:

$2x^2 m(m-1)x^{m-2} + 5x\, mx^{m-1} + x^m = 0$

$x^2\left[2m^2 + 3m + 1\right] = 0$

$m = \dfrac{-3 \pm \sqrt{9-4(2)(1)}}{2(2)} = \dfrac{-3 \pm 1}{4} = \frac{-1}{2}, -1$

[4]

$\boxed{\therefore\ y = c_1 x^{-1/2} + c_2 x^{-1}}$

Question 3:

(a) Solve the following initial value problem:

$$y'' - 5y' + 4y = 0, \qquad y(0) = 12, y'(0) = -3$$

$r^2 - 5r + 4 = 0$

$(r-1)(r-4) = 0 \Rightarrow r = 1, 4$

$y = c_1 e^{t} + c_2 e^{4t}$

$y(0) = c_1 + c_2 = 12 \Rightarrow ① \; c_1 + c_2 = 12$

$y'(0) = c_1 e^{t} + 4c_2 e^{4t}\Big|_{t=0} = -3 \Rightarrow ② \; c_1 + 4c_2 = -3$

② − ①: $3c_2 = -15$

$c_2 = -5$

$\therefore c_1 = 12 - (-5) = 17$

$\boxed{\therefore \; y = 17e^{t} - 5e^{4t}}$

[5]

(b) Use part 2(b) to find the general solution of

$$y'' + 2y' + y = e^{-x} \ln(x)$$

Using variation of parameters with $y_1 = e^{-x}$, $y_2 = xe^{-x}$:

$W = \begin{vmatrix} e^{-x} & xe^{-x} \\ -e^{-x} & e^{-x} - xe^{-x} \end{vmatrix} = e^{-2x} - x\cancel{e^{-2x}} + x\cancel{e^{-2x}} = e^{-2x}$

$u_1' = -\dfrac{y_2 f(x)}{W} = -\dfrac{xe^{-x} \cdot e^{-x} \ln(x)}{e^{-2x}} = -x\ln(x)$

$\therefore u_1 = -\int x \ln(x)\, dx = -\int \ln(x)\, d\left[\frac{x^2}{2}\right] = -\dfrac{x^2 \ln(x)}{2} + \int \frac{x^2}{2} \cdot \frac{1}{x}\, dx = -\dfrac{x^2\ln(x)}{2} + \dfrac{x^2}{4}$

$u_2' = \dfrac{y_1 f(x)}{W} = \dfrac{e^{-x}\, e^{-x} \ln(x)}{e^{-2x}} = \ln(x)$

$\therefore u_2 = \int \ln(x)\, dx = x\ln(x) - x$

$\therefore y_p = u_1 y_1 + u_2 y_2 = -\dfrac{x^2 e^{-x} \ln(x)}{2} + \dfrac{x^2 e^{-x}}{4} + x^2 e^{-x} \ln(x) \underbrace{- xe^{-x}}_{\text{part of } y_c}$

$= \dfrac{x^2 e^{-x} \ln(x)}{2} + x^2 e^{-x}/4$

[5]

$\boxed{\therefore y = y_c + y_p = c_1 e^{-x} + c_2 x e^{-x} + \dfrac{x^2 e^{-x} \ln(x)}{2} + \dfrac{x^2 e^{-x}}{4}}$

Question 4: A 1 kg mass is attached to a spring whose constant is $k = 16$ N/m. The entire spring-mass system is then submerged in a liquid that imparts a damping force of $\beta = 10$ times the velocity of the mass. At time $t = 0$ the mass is released from a point 1 m below equilibrium with an initial velocity of 12 m/s upward. Set up and solve the differential equation describing the position $x(t)$ of the mass for $t \geq 0$.

$m=1$, $k=16$, $\beta = 10x'$, $x(0) = 1$, $x'(0) = -12$

$$mx'' = -kx - \beta x'$$

$$x'' + 10x' + 16x = 0$$

$$r^2 + 10r + 16 = 0$$

$$(r+8)(r+2) = 0$$

$$r = -8,\ r = -2$$

$$\therefore\ x = c_1 e^{-2t} + c_2 e^{-8t}$$

$$x(0) = 1 \Rightarrow c_1 + c_2 = 1 \quad (1)$$

$$x'(0) = -12 \Rightarrow -2c_1 e^{-2t} - 8c_2 e^{-8t}\Big|_{t=0} = -12$$

$$\Rightarrow -2c_1 - 8c_2 = -12$$

$$\Rightarrow c_1 + 4c_2 = 6 \quad (2)$$

$$(2) - (1):\ 3c_2 = 5 \Rightarrow c_2 = \frac{5}{3}$$

$$\therefore\ c_1 = 1 - c_2 = 1 - \frac{5}{3} = -\frac{2}{3}$$

$$\boxed{\therefore\ x(t) = -\frac{2}{3}e^{-2t} + \frac{5}{3}e^{-8t}}$$

[10]

Question 5: The temperature $u(r)$ between concentric spheres of radius $r = a$ and $r = b$, (where $a < b$) is determined by the boundary value problem

$$r\frac{d^2u}{dr^2} + 2\frac{du}{dr} = 0, \qquad u(a) = u_0, u(b) = u_1,$$

where u_0 and u_1 are constants. Solve for $u(r)$.

$$ru'' + 2u' = 0 \quad , \quad u(a) = u_0 \; , \; u(b) = u_1$$

$$r^2u'' + 2ru' = 0$$

$$u = r^m \; , \; u' = mr^{m-1} \; , \; u'' = m(m-1)\,r^{m-2}$$

$$r^2\left[m(m-1)r^{m-2}\right] + 2r\left[mr^{m-1}\right] = 0$$

$$r^m\left[m^2 + m\right] = 0$$

$$m(m+1) = 0$$

$$m = 0, \; -1$$

$$\therefore \; u = c_1 + \frac{c_2}{r}$$

$$u(a) = u_0 \implies c_1 + \frac{c_2}{a} = u_0 \quad ①$$

$$u(b) = u_1 \implies c_1 + \frac{c_2}{b} = u_1 \quad ②$$

$$② - ① : \; c_2\left(\frac{1}{b} - \frac{1}{a}\right) = u_1 - u_0 \implies c_2 = \frac{ab(u_1 - u_0)}{(a-b)}$$

$$\therefore \; c_1 = u_0 - \frac{1}{a}c_2 = u_0 - \frac{1}{\cancel{a}} \cdot \frac{\cancel{a}b(u_1 - u_0)}{(a-b)} = \frac{au_0 - bu_1}{(a-b)}$$

$$\boxed{\therefore \; u(r) = \left(\frac{au_0 - bu_1}{a-b}\right) + \left[\frac{ab(u_1 - u_0)}{a-b}\right]\left(\frac{1}{r}\right)}$$

[10]

Question 6:

(a) Calculate $\mathcal{L}\{f(t)\}$ where $f(t) = te^{-2t-3}$ (do not use tables)

$$\mathcal{L}\{te^{-2t-3}\} = \int_0^\infty e^{-st}\, t e^{-2t-3}\, dt$$
$$= e^{-3}\int_0^\infty t\, e^{-(s+2)t}\, dt$$
$$= e^{-3}\int_0^\infty t\, d\left[\frac{e^{-(s+2)t}}{-(s+2)}\right]$$
$$= \frac{e^{-3}}{-(s+2)}\left[t e^{-(s+2)t}\Big|_0^\infty - \int_0^\infty e^{-(s+2)t}\, dt\right]$$
$$= \frac{-e^{-3}}{(s+2)^2}\left[e^{-(s+2)t}\right]_0^\infty$$
$$= \boxed{\frac{e^{-3}}{(s+2)^2}}$$

[3]

(b) Determine $\mathcal{L}^{-1}\{F(s)\}$ where $F(s) = (s^2 + s - 20)^{-1}$

$$F(s) = \frac{1}{s^2+s-20} = \frac{1}{(s-4)(s+5)}, \quad \therefore \mathcal{L}^{-1}\{F(s)\} = \boxed{\frac{e^{4t}-e^{-5t}}{4-(-5)} = \frac{e^{4t}-e^{-5t}}{9}}$$

(using formula #28).

Or, by partial fractions:

$$\frac{1}{(s-4)(s+5)} = \frac{(1/9)}{s-4} + \frac{(-1/9)}{s+5} \Rightarrow \mathcal{L}^{-1}\{F(s)\} = \frac{1}{9}\mathcal{L}^{-1}\left\{\frac{1}{s-4}\right\} - \frac{1}{9}\mathcal{L}^{-1}\left\{\frac{1}{s+5}\right\} =$$

[3]

(c) Solve using the Laplace transform:

$$y'' + 9y = e^t, \quad y(0) = 0, \quad y'(0) = 0$$

$$\mathcal{L}\{y'' + 9y\} = \mathcal{L}\{e^t\}$$
$$s^2 Y - s y(0) - y'(0) + 9Y = \frac{1}{s-1}$$
$$Y(s^2+3^2) = \frac{1}{s-1}$$
$$Y = \frac{1}{(s-1)(s^2+3^2)} = \frac{A}{s-1} + \frac{Bs+C}{s^2+3^2}$$
$$= \frac{(A+B)s^2 + (-B+C)s + (9A-C)}{(s-1)(s^2+3^2)}$$

$$\therefore \left.\begin{aligned} A+B&=0 \quad ① \\ -B+C&=0 \quad ② \\ 9A-C&=1 \quad ③ \end{aligned}\right\} \quad \left.\begin{aligned} ① &\Rightarrow A = -B \\ ② &\Rightarrow C = B = -A \\ ③ &\Rightarrow 9A-(-A)=1 \end{aligned}\right\} \quad \begin{aligned} \therefore A &= \tfrac{1}{10} \\ B &= \tfrac{-1}{10} \\ C &= -\tfrac{1}{10} \end{aligned}$$

$$\therefore Y = \frac{1}{10}\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} - \frac{1}{10}\mathcal{L}^{-1}\left\{\frac{s}{s^2+3^2}\right\} - \frac{1}{30}\left\{\frac{3}{s^2+3^2}\right\} = \boxed{\frac{e^t}{10} - \frac{\cos(3t)}{10} - \frac{\sin(3t)}{30}}$$

[4]