Real Vector Spaces: Overview

Consider $R^3 = \{(v_1, v_2, v_3) \mid v_i \in \mathbb{R}\}$.

- We saw that R^3 consists of all **linear combinations**

  $$a\,\hat{\mathbf{i}} + b\,\hat{\mathbf{j}} + c\,\hat{\mathbf{k}}$$

  where a, b and c are scalars.

- R^3 is an example of a **vector space** over $\mathbb{R}$. We say that R^3 is **spanned** by $\left\{\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}\right\}$.

- For any non-zero vector $\mathbf{u}$ we saw that

  $$W_1 = \{a\,\mathbf{u} \mid a \in \mathbb{R}\}$$

  is a line through $\mathbf{0}$. Sums and scalar multiples of vectors in W_1 are again in the line W_1.

- For any non-zero vectors $\mathbf{u}$ and $\mathbf{w}$ we saw that

  $$W_2 = \{a\,\mathbf{u} + b\,\mathbf{w} \mid a, b \in \mathbb{R}\}$$

  is a plane through $\mathbf{0}$. Sums and scalar multiples of vectors in W_2 are again in the plane W_2.

- W_1 and W_2 are called **subspaces** of R^3 .

These ideas are useful for studying other mathematical objects which can can also be viewed as vectors.

Definition of a Vector Space

Let V be a nonempty set of objects (called vectors) on which two operations are defined: addition and scalar multiplication.

Addition assigns to each pair of vectors $\mathbf{u}$ and $\mathbf{v}$ a vector denoted $\mathbf{u}+\mathbf{v}$.

Scalar multiplication assigns to each scalar k and vector $\mathbf{u}$ a vector denoted $k\mathbf{u}$.

If for all vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ and scalars k and m the following **axioms** (basic assumptions) are satisfied then we say that V is a **vector space**:

1. $\mathbf{u}+\mathbf{v}$ is in V. (Say that V is **closed** under addition.)
2. $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$
3. $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$
4. There is an object $\mathbf{0}$ in V called the zero object such that $\mathbf{0}+\mathbf{u}=\mathbf{u}+\mathbf{0}$ for every u in V .
5. For each $\mathbf{u}$ in V there is $-\mathbf{u}$ such that $\mathbf{u}+(-\mathbf{u})=(-\mathbf{u})+\mathbf{u}=\mathbf{0}$.
6. $k\mathbf{u}$ is in V for all scalars k. (Say that V is **closed** under scalar multiplication.)
7. $k(\mathbf{u}+\mathbf{v})=k\mathbf{u}+k\mathbf{v}$
8. $(k+m)\mathbf{u}=k\mathbf{u}+m\mathbf{u}$
9. $k(m\mathbf{u})=(km)\mathbf{u}$
10. $1\mathbf{u}=\mathbf{u}$

For us the set of scalars will be the real numbers, in which case V is called **a real vector space**.

Examples of Vector Spaces

1. R^n with the usual addition and scalar multiplication of vectors.

2. $R^\infty =$ the set of all infinite sequences of real numbers $\mathbf{u} = (u_1, u_2, u_3, \dots)$ with addition and scalar multiplication defined component-wise.

3. $M_{mn} =$ the set of all $m \times n$ matrices with matrix addition and scalar multiplication.

4. $F(-\infty, \infty) =$ the set of all real valued functions with domain $(-\infty, \infty)$:

   Suppose f, g are functions with domain $(-\infty, \infty)$ and a, b are real scalars. Then $af + bg$ defined by

   $$(af + bg)(x) = af(x) + bg(x)$$

   again has domain $(-\infty, \infty)$.

   Here the vectors are functions.

5. $C(-\infty, \infty) =$ the set of all real valued continuous functions with domain $(-\infty, \infty)$.

   This time if f, g are continuous functions with domain $(-\infty, \infty)$ and a, b are real scalars, then $af + bg$ is again continuous with domain $(-\infty, \infty)$.

6. $C^m(-\infty, \infty) =$ the set of all real valued functions whose first m derivatives exist and are continuous on $(-\infty, \infty)$.

7. $P_n =$ the set of all polynomials of degree less than or equal to n:

   $$p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$$

   where $a_0, a_1, a_2, \dots, a_n$ are real numbers.