Question 1: Use the Wronskian to determine if $S = \{x, e^{-x}, xe^{x}\}$ is a linearly independent set on $(-\infty, \infty)$.

$y_1 = x,\ y_1' = 1,\ y_1'' = 0$

$y_2 = e^{-x},\ y_2' = -e^{-x},\ y_2'' = e^{-x}$

$y_3 = xe^{x},\ y_3' = e^{x} + xe^{x},\ y_3'' = 2e^{x} + xe^{x}$

$$W = \begin{vmatrix} x & e^{-x} & xe^{x} \\ 1 & -e^{-x} & e^{x}+xe^{x} \\ 0 & e^{-x} & 2e^{x}+xe^{x} \end{vmatrix}$$

$$= x\left[(-2-x)-(1+x)\right] - \left[(2+x)-x\right]$$

$$= -2x^2 - 3x - 2$$

$$= -(2x^2 + 3x + 2)$$

$\neq 0$ on $(-\infty, \infty)$ since $b^2 - 4ac = 3^2 - 4(2)(2) < 0$

$\therefore S$ is linearly independent.

[5]

Question 2: Determine whether $S = \{(1,0,0,0), (1,1,1,1), (1,0,1,0), (0,1,0,1)\}$ is a basis for $\mathbb{R}^4$.

Since S contains 4 vectors and $\dim(\mathbb{R}^4) = 4$, S is a basis if and only if

$$\begin{vmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{vmatrix} \neq 0$$

using col. 1

$$= 1\left[(1)(1-0) - 0(\sim) + 1(0-1)\right]$$

$$= 0$$

$\therefore S$ is <u>not</u> a basis for $\mathbb{R}^4$.

[5]

Question 3: Let $\mathbf{p}_1 = 1 + x$, $\mathbf{p}_2 = 1 + x^2$ and $\mathbf{p}_3 = x + x^2$. If $\mathbf{p} = 2 - x + x^2$ and $S = \{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ is a basis, find $(\mathbf{p})_S$. (Recall, $(\mathbf{p})_S$ is the coordinate vector for $\mathbf{p}$ relative to the basis S.)

$(\vec{p})_S = (a, b, c)$ where $a\vec{p_1} + b\vec{p_2} + c\vec{p_3} = \vec{p}$, so solve

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 0 & 2 \\ 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

$R_2 = (-1)r_1 + r_2$

$$\begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & -1 & 1 & -3 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

$R_2 = (-1)r_2$:

$$\begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 1 & -1 & 3 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

$R_3 = (-1)r_2 + r_3$:

$$\begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 2 & -2 \end{bmatrix}$$

$R_3 = \frac{1}{2}r_3$:

$$\begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

$\therefore c = 1$

$b = 3 + c = 2$

$a = 2 - b = 0$

$\therefore (p)_S = (0, 2, -1)$

[5]

Question 4: Find a basis and state the dimension of the row space of

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 2 & -2 \\ 4 & 3 & -1 \\ 6 & 5 & 1 \end{bmatrix}$$

$$\left.\begin{array}{l} R_2 = (-3)r_1 + r_2 \\ R_3 = (-4)r_1 + r_3 \\ R_4 = (-6)r_1 + r_4 \end{array}\right\} \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -5 \\ 0 & -1 & -5 \\ 0 & -1 & -5 \end{bmatrix}$$

$R_2 = (-1)r_2$:

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 5 \\ 0 & -1 & -5 \\ 0 & -1 & -5 \end{bmatrix}$$

$$\left.\begin{array}{l} R_3 = (-1)r_2 + r_3 \\ R_4 = (-1)r_2 + r_4 \end{array}\right\} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ Basis is $\{(1,1,1), (0,1,5)\}$

dimension is 2.

[5]

Question 5: Find a basis and state the dimension for the subspace of $\mathbb{R}^4$ consisting of all vectors (a, b, c, d) where $b = a + c$ and $d = -2a + 3c$.

Subspace consists of all vectors of the form $\begin{bmatrix} a \\ a+c \\ c \\ -2a+3c \end{bmatrix} = a\begin{bmatrix} 1 \\ 1 \\ 0 \\ -2 \end{bmatrix} + c\begin{bmatrix} 0 \\ 1 \\ 1 \\ 3 \end{bmatrix}$

$\therefore$ Basis is $\{(1,1,0,-2), (0,1,1,3)\}$, dimension is 2.

[5]

Question 6: Find a basis for the subspace of R^3 spanned by $\mathbf{v}_1 = (1, -1, 3)$, $\mathbf{v}_2 = (5, -4, -4)$ and $\mathbf{v}_3 = (7, -6, 2)$.

$$\begin{bmatrix} 1 & -1 & 3 \\ 5 & -4 & -4 \\ 7 & -6 & 2 \end{bmatrix}$$

$R_2 = (-5)r_1 + r_2$:
$R_3 = (-7)r_1 + r_3$:

$$\begin{bmatrix} 1 & -1 & 3 \\ 0 & 1 & -19 \\ 0 & 1 & -19 \end{bmatrix}$$

$R_3 = (-1)r_2 + r_3$:

$$\begin{bmatrix} 1 & -1 & 3 \\ 0 & 1 & -19 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ Basis is $\{(1,-1,3), (0,1,-19)\}$

[5]

Question 7: The matrix $\mathbf{A} = \begin{bmatrix} 1 & 3 & 1 & 3 \\ 0 & 1 & 1 & 0 \\ -3 & 0 & 6 & -1 \\ 3 & 4 & -2 & 1 \\ 2 & 0 & -4 & -2 \end{bmatrix}$ has REF $\mathbf{R} = \begin{bmatrix} 1 & 3 & 1 & 3 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

(a) Give a basis and state the dimension of the row space of **A** .

$\{(1,3,1,3), (0,1,1,0), (0,0,0,1)\}$; dimension 3.

[2]

(b) Give a basis and state the dimension of the column space of **A** .

$\{(1,0,-3,3,2), (3,1,0,4,0), (3,0,-1,1,-2)\}$; dimension 3

[2]

(c) Give a basis and state the dimension of the null space of **A** . (Equivalently, give a basis and state the dimension of the solution space of $\mathbf{Ax} = \mathbf{0}$.)

Let $x_3 = r$, from R: $x_4 = 0$; $x_2 = -x_3 = -r$;

$x_1 = -3x_4 - x_3 - 3x_2 = 2r$

$\therefore$ Solutions have form $\begin{bmatrix} 2r \\ -r \\ r \\ 0 \end{bmatrix} = r\begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix}$, so

null space has basis $\{(2,-1,1,0)\}$; dimension is 1. [3]

(d) Determine rank(**A**) and nullity(**A**) .

rank(A) = 3, nullity(A) = 1

[1]

(e) Determine rank($\mathbf{A}^T$) and nullity($\mathbf{A}^T$) .

rank(A^T) = 3, nullity(A^T) = 2.

[1]

Question 8: Let the transformation T_1 represent projection onto the xy-plane and T_2 be the transformation which rotates a vector in R^3 counter-clockwise about the y-axis by an angle θ.

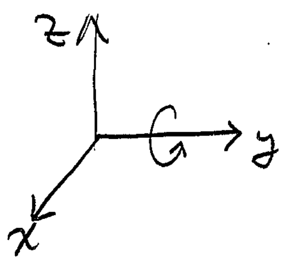
(a) Determine the standard matrix **A** for the transformation T_1 .

$$T_1\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = \begin{bmatrix} a \\ b \\ 0 \end{bmatrix} \Rightarrow T_1\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix};\ T_1\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix};\ T_1\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

[2]

(b) Determine the standard matrix **B** for the transformation T_2 .



$$T_2\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} \cos\theta \\ 0 \\ -\sin\theta \end{bmatrix} \quad \therefore B = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}.$$

$$T_2\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$T_2\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} \sin\theta \\ 0 \\ \cos\theta \end{bmatrix}$$

[3]

(c) Determine the image of the line segment joining $(2, 1, 1)$ and $(1, 3, -1)$ if it is first rotated counter-clockwise about the y-axis by $\pi/3$ (or $60°$) and then projected onto the xy-plane.

$$AB\left[\begin{array}{c|c} 2 & 1 \\ 1 & 3 \\ 1 & -1 \end{array}\right]$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \frac{1}{2} & 0 & \frac{\sqrt{3}}{2} \\ 0 & 1 & 0 \\ -\frac{\sqrt{3}}{2} & 0 & \frac{1}{2} \end{bmatrix}\left[\begin{array}{c|c} 2 & 1 \\ 1 & 3 \\ 1 & -1 \end{array}\right]$$

$$= \begin{bmatrix} \frac{1}{2} & 0 & \frac{\sqrt{3}}{2} \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\left[\begin{array}{c|c} 2 & 1 \\ 1 & 3 \\ 1 & -1 \end{array}\right]$$

$$= \left[\begin{array}{c|c} \frac{2+\sqrt{3}}{2} & \frac{1-\sqrt{3}}{2} \\ 1 & 3 \\ 0 & 0 \end{array}\right]$$

$\therefore$ image is line segment from $\left(\frac{2+\sqrt{3}}{2}, 1, 0\right)$ to $\left(\frac{1-\sqrt{3}}{2}, 3, 0\right)$

[3]

(d) Is applying the projection followed by the rotation the same as applying the rotation followed by the projection? Explain using matrices.

No. Using $\theta = \pi/3$ as above,

$$AB = \begin{bmatrix} \frac{1}{2} & 0 & \frac{\sqrt{3}}{2} \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \text{ but } BA = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{\sqrt{3}}{2} & 0 & 0 \end{bmatrix}$$

[2]

So $AB \neq BA$.