Question 1: Solve the following system of equations **using either Gaussian or Gauss-Jordan elimination** (no credit will be given for using any other method). Use proper notation to clearly state the row operations used at each step and clearly state the final solution.

$$2x - 6z = -4$$
$$3x + y - 2z = 5$$
$$2x + 2y + z = 4$$

$$\begin{bmatrix} 2 & 0 & -6 & -4 \\ 3 & 1 & -2 & 5 \\ 2 & 2 & 1 & 4 \end{bmatrix}$$

$R_1 = \frac{1}{2} r_1$:

$$\begin{bmatrix} 1 & 0 & -3 & -2 \\ 3 & 1 & -2 & 5 \\ 2 & 2 & 1 & 4 \end{bmatrix}$$

$R_2 = (-3) r_1 + r_2$:

$R_3 = (-2) r_1 + r_3$:

$$\begin{bmatrix} 1 & 0 & -3 & -2 \\ 0 & 1 & 7 & 11 \\ 0 & 2 & 7 & 8 \end{bmatrix}$$

$R_3 = (-2) r_2 + r_3$:

$$\begin{bmatrix} 1 & 0 & -3 & -2 \\ 0 & 1 & 7 & 11 \\ 0 & 0 & -7 & -14 \end{bmatrix}$$

$R_3 = \frac{-1}{7} r_3$:

$$\begin{bmatrix} 1 & 0 & -3 & -2 \\ 0 & 1 & 7 & 11 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$R_1 = 3r_3 + r_1$:

$R_2 = (-7) r_3 + r_1$:

$$\begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$\boxed{\therefore\ x = 4,\ y = -3,\ z = 2}$

[10]

Question 2: Solve the following system of equations **using either Gaussian or Gauss-Jordan elimination** (no credit will be given for using any other method). Use proper notation to clearly state the row operations used at each step and clearly state the final solution.

$$2x + 3z = 3$$
$$4x - 3y + 7z = 5$$
$$8x - 9y + 15z = 10$$

$$\begin{bmatrix} 2 & 0 & 3 & 3 \\ 4 & -3 & 7 & 5 \\ 8 & -9 & 15 & 10 \end{bmatrix}$$

$R_1 = \frac{1}{2} r_1$:

$$\begin{bmatrix} 1 & 0 & \frac{3}{2} & \frac{3}{2} \\ 4 & -3 & 7 & 5 \\ 8 & -9 & 15 & 10 \end{bmatrix}$$

$R_2 = (-4) r_1 + r_2$:

$R_3 = (-8) r_1 + r_3$:

$$\begin{bmatrix} 1 & 0 & 3/2 & 3/2 \\ 0 & -3 & 1 & -1 \\ 0 & -9 & 3 & -2 \end{bmatrix}$$

$R_3 = (-3) r_2 + r_3$:

$$\begin{bmatrix} 1 & 0 & 3/2 & 3/2 \\ 0 & -3 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

no solution!

system is inconsistent.

[10]

Question 3: Solve the following system of equations **using either Gaussian or Gauss-Jordan elimination** (no credit will be given for using any other method). Use proper notation to clearly state the row operations used at each step and clearly state the final solution.

$$\begin{aligned} 3x + 3y + 12z &= 6 \\ x + y + 4z &= 2 \\ 2x + 5y + 20z &= 10 \\ -x + 2y + 8z &= 4 \end{aligned}$$

$$\begin{bmatrix} 3 & 3 & 12 & 6 \\ 1 & 1 & 4 & 2 \\ 2 & 5 & 20 & 10 \\ -1 & 2 & 8 & 4 \end{bmatrix}$$

$r_1 \leftrightarrow r_2$:

$$\begin{bmatrix} 1 & 1 & 4 & 2 \\ 3 & 3 & 12 & 6 \\ 2 & 5 & 20 & 10 \\ -1 & 2 & 8 & 4 \end{bmatrix}$$

$R_2 = (-3)r_1 + r_2$;
$R_3 = (-2)r_1 + r_3$;
$R_4 = (1)r_1 + r_4$:

$$\begin{bmatrix} 1 & 1 & 4 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 3 & 12 & 6 \\ 0 & 3 & 12 & 6 \end{bmatrix}$$

$R_4 = (-1)r_3 + r_4$:

$$\begin{bmatrix} 1 & 1 & 4 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 3 & 12 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$r_2 \leftrightarrow r_3$:

$$\begin{bmatrix} 1 & 1 & 4 & 2 \\ 0 & 3 & 12 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_2 = \frac{1}{3} r_2$;

$$\begin{bmatrix} 1 & 1 & 4 & 2 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_1 = (-1)r_2 + r_1$:

$$\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore\ z = t$
$y = 3 - 4t$
$x = -1$

$\therefore (x, y, z) = (-1, 3-4t, t)$ where t is any real number

[10]

Question 4: Use matrices to determine the values of a, b and c so that the polynomial $y = ax^2 + bx + c$ passes through the points $(1,4)$, $(2,0)$ and $(3,12)$.

$$4 = a(1)^2 + b(1) + c$$
$$0 = a(2)^2 + b(2) + c$$
$$12 = a(3)^2 + b(3) + c$$

$$\therefore \begin{bmatrix} 1 & 1 & 1 & 4 \\ 4 & 2 & 1 & 0 \\ 9 & 3 & 1 & 12 \end{bmatrix}$$

$R_2 = (-4)r_1 + r_2$:

$R_3 = (-9)r_1 + r_3$:

$$\begin{bmatrix} 1 & 1 & 1 & 4 \\ 0 & -2 & -3 & -16 \\ 0 & -6 & -8 & -24 \end{bmatrix}$$

$R_3 = (-3)r_2 + r_3$:

$$\begin{bmatrix} 1 & 1 & 1 & 4 \\ 0 & -2 & -3 & -16 \\ 0 & 0 & 1 & 24 \end{bmatrix}$$

$R_2 = \frac{-1}{2} r_2$:

$$\begin{bmatrix} 1 & 1 & 1 & 4 \\ 0 & 1 & 3/2 & 8 \\ 0 & 0 & 1 & 24 \end{bmatrix}$$

$\therefore c = 24,$

$b = 8 - \frac{3}{2}(z) = 8 - \frac{3}{2}(24) = -28$

$a = 4 - y - z = 4 + 28 - 24 = 8$

$\therefore a = 8, \quad b = -28, \quad c = 24$

[5]

Question 5: Determine all values of a (if any) for which the following system has exactly one solution:

$$x + 2y = 1$$
$$2x + (a^2 - 5)y = a - 1$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & (a^2-5) & (a-1) \end{bmatrix}$$

$R_2 = (-2)r_1 + r_2$:

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & a^2-9 & a-3 \end{bmatrix}$$

notice: $\begin{cases} a \neq 3, & \text{otherwise infinitely many sol}^{n}\text{s.} \\ a \neq -3, & \text{otherwise no solution.} \end{cases}$

$R_2 = \left(\frac{1}{a^2-9}\right) r_2$:

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & \frac{a-3}{a^2-9} \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & \frac{1}{a+3} \end{bmatrix}$$

$\therefore$ For $a \neq 3$, $a \neq -3$ we have the single solution $y = \frac{1}{a+3}$, $x = 1 - 2\left(\frac{1}{a+3}\right)$.

[5]

Question 6: For this problem use the following matrices to carry out the indicated computations, if possible. If a given statement is not defined then state "not defined":

$$\mathbf{A} = \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 4 & -1 \\ 1 & 2 \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix} \quad \mathbf{D} = \begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix} \quad \mathbf{E} = \begin{bmatrix} 6 & 1 & 3 \\ -1 & 1 & 2 \\ 4 & 1 & 3 \end{bmatrix}$$

(a) $\mathbf{BC} - \mathbf{A}^{\mathsf{T}}$

$\underbrace{2\times 2 \quad 2\times 3}_{2\times 3} \quad 2\times 3$ ✓

$$= \begin{bmatrix} 4 & -1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix} - \begin{bmatrix} 3 & -1 & 1 \\ 0 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 15 & 3 \\ 7 & 6 & 12 \end{bmatrix} - \begin{bmatrix} 3 & -1 & 1 \\ 0 & 2 & 1 \end{bmatrix} = \boxed{\begin{bmatrix} -2 & 16 & 2 \\ 7 & 4 & 11 \end{bmatrix}}$$

[2]

(b) $\operatorname{tr}(\mathbf{AC} - 4\mathbf{D}^{\mathsf{T}})$

$\underbrace{3\times 2 \quad 2\times 3}_{3\times 3} \quad 3\times 3$ ✓

$$= tr\left(\begin{bmatrix} 3 & & \\ & -2 & \\ & & 7 \end{bmatrix} - 4\begin{bmatrix} 1 & & \\ & 0 & \\ & & 4 \end{bmatrix}\right)$$

$$= tr\begin{bmatrix} -1 & & \\ & -2 & \\ & & -9 \end{bmatrix} = \boxed{-12}$$

[2]

(c) $7\mathbf{DE} + 5\mathbf{CA}$

$3\times 3 \quad \underbrace{2\times 3 \quad 3\times 2}_{2\times 2}$

X

NOT DEFINED!

[2]

(d) Determine the size of **ABCDE**

$3\times 2 \quad 2\times 2 \quad 2\times 3 \quad 3\times 3 \quad 3\times 3$ ✓ ✓ ✓ ✓

$\boxed{3\times 3}$

[2]

(e) Find a matrix $\begin{bmatrix} x \\ y \end{bmatrix}$ so that $\mathbf{B}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ -2 \end{bmatrix}$.

$$\begin{bmatrix} 4 & -1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ -2 \end{bmatrix} \Rightarrow \begin{array}{l} 4x - y = 7 \\ x + 2y = -2 \end{array}$$

$$\therefore \begin{bmatrix} 4 & -1 & 7 \\ 1 & 2 & -2 \end{bmatrix} \xrightarrow{r_1 \leftrightarrow r_2} \begin{bmatrix} 1 & 2 & -2 \\ 4 & -1 & 7 \end{bmatrix} \xrightarrow{R_2 = (-4)r_1 + r_2} \begin{bmatrix} 1 & 2 & -2 \\ 0 & -9 & 15 \end{bmatrix} \xrightarrow{R_2 = (\frac{-1}{9})r_2} \begin{bmatrix} 1 & 2 & -2 \\ 0 & 1 & -\frac{5}{3} \end{bmatrix}$$

[2]

$$\therefore y = -\frac{5}{3}, \quad x = -2 - 2\left(\frac{-5}{3}\right) = \frac{4}{3} \qquad \therefore \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4/3 \\ -5/3 \end{bmatrix}$$