Math 141 - Matrix Algebra for Engineers

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Systems of Linear Equations

Goal

We wish to solve systems of linear equations, for example

$$
\begin{aligned}
4x + 3y - 6z &= -2 \\
-x + y + z &= 2 \\
x - 2y - z &= -3
\end{aligned}
$$

That is, we wish to find all (x, y, z) which satisfy all equations simultaneously.

First, some terminology and notation...

Linear Equations

A **linear equation in *n*-variables** $x_1, x_2, \ldots, x_n$ is an equation of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

where $a_1, a_2, \ldots, a_n$ and b are real numbers.

Example: $2x + 4y - z = 5$ is a linear equation in 3-variables.

If $b = 0$, say the equation is **homogeneous**.

Example: $7x_1 + 3x_2 - 4x_3 + 11x_4 = 0$ is a homogeneous linear equation in 4-variables.

What Makes a Linear Equation Linear

Important: in a linear equation, each term involving the variable is of form

(real number)(variable raised to power 1)

There are no terms of form

$$\sin x, \; \cos y, \; \ln x, \; e^x, \; xy, \; 1/x, \; x^2, \; \sqrt{1+x}, \; \text{etc}$$

Systems of Linear Equations

A **system of linear equations** is a finite collection of linear equations:

Example: A system of 2 equations in 3 variables, or 3 **unknowns**:

$$\begin{aligned} 4x + 3y - 6z &= -2 \\ -x + y + z &= 2 \end{aligned}$$

A General System

A general system of m equations in n unknowns:

$$\begin{array}{ccccccccc}
a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\
a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\
\vdots & & \vdots & & & & \vdots & & \\
a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m
\end{array}$$

A **solution** to this system is a set of numbers $x_1 = p_1, x_2 = p_2, \ldots, x_n = p_n$ which make all equations of the system true simultaneously.

Sometimes solutions are stated in **vector** form $(p_1, p_2, \ldots, p_n)$, also called an **ordered n-tuple**.

Consistent Systems of Equations

A system of linear equations is called **consistent** if it has **at least one solution**, otherwise it is called **inconsistent**.

Example: We saw that

$$\begin{aligned} 2x + 3y &= -4 \\ -3x + y &= -5 \end{aligned}$$

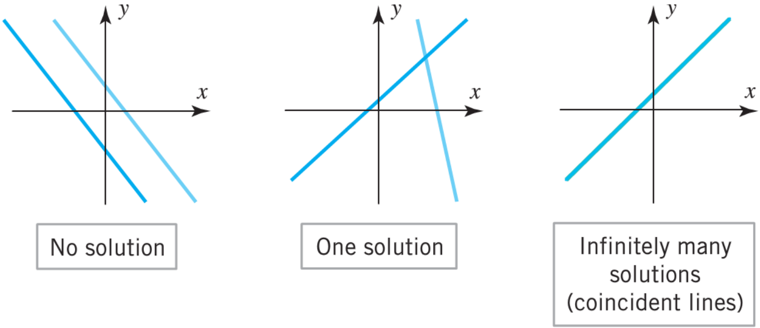
had solution $(1, -2)$, so the system is consistent.

Geometrically this says that the two lines intersect at the point $(1, -2)$.

Two Equations in Two Unknowns

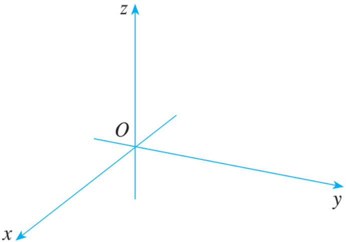
A linear equation in 2-variables represents a line in 2-dimensions.

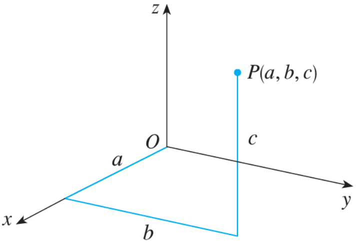
Two equations in two unknowns corresponds to lines that are either parallel, intersecting, or coincident:



Linear Equations in Three Variables

To graph an equation in 3-variables requires the three dimensional rectangular coordinate system:





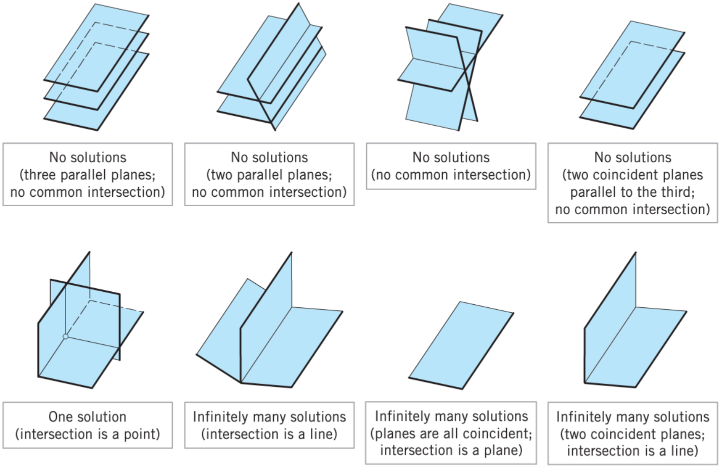
Planes in 3-Dimensions

The graph (set of all points satisfying the equation) of a linear equation in 3-variables is a **plane**, an infinite flat sheet.

Example: Graph the plane $x + 2y + 3z = 6$

Three Equations in Three Unknowns

The solution of a system of three equations in three unknowns describes how the three planes intersect in three dimensions:



Important Observation

Notice that in the case of linear systems in two or three variables, **the systems have either zero, exactly one, or infinitely many solutions**. This turns out to be true for any system of linear equations.

Representing Infinitely Many Solutions

Example: Solve the system

$$\begin{aligned} x - 2y &= 4 \\ -\frac{1}{4}x + \frac{1}{2}y &= -1 \end{aligned}$$

Solution: (one way, to illustrate a point) Multiply the second equation by 4:

$$\begin{aligned} x - 2y &= 4 \\ -x + 2y &= -4 \end{aligned}$$

Now add the first equation to the second:

$$\begin{aligned} x - 2y &= 4 \\ 0 &= 0 \end{aligned}$$

Representing Infinitely Many Solutions

This says that any (x, y) which satisfies $x - 2y = 4$ is a solution to the system. There are infinitely many such pairs which we represent in **parametric form**:

Since $x - 2y = 4, \quad x = 4 + 2y$.

Let $y = t$, so $x = 4 + 2t$.

So the solution to the system is $(4 + 2t, t)$ where t is any real number.

Parametric Representation of Solutions

In the solution $(4 + 2t, t)$ the variable t is called a **parameter**. Letting the parameter vary over all real numbers produces all possible solutions.

For example, letting $t = 0$ gives the solution $(4, 0)$.

Letting $t = -3$ gives another solution $(-2, -3)$.

Letting $t = \pi$ gives $(4 + 2\pi, \pi)$, etc.

Another Example: Infinitely Many Solutions

Example: Solve the system

$$3x + 2y - z = 3$$

Solution: This time, any (x, y, z) which satisfies the single equation is a solution. Once any two of the variables are specified, the third is determined, so two parameters are needed to represent the solution set.

Write $x = 1 - \frac{2}{3}y + \frac{1}{3}z$.

Let $y = r$, $z = t$, so that $x = 1 - \frac{2}{3}r + \frac{1}{3}t$.

The solution is then $(1 - \frac{2}{3}r + \frac{1}{3}t, r, t)$ where r and t are any real numbers.