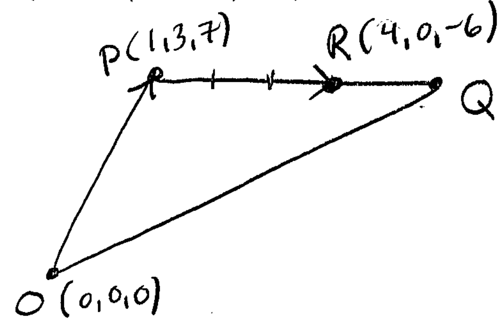
Question 1: Let P be the point $(1, 3, 7)$. Determine the point Q if the point $R(4, 0, -6)$ is 3/4 of the distance from P to Q.



$$\vec{OQ} = \vec{OP} + \vec{PR} + \tfrac{1}{3}\vec{PR}$$
$$= \vec{OP} + \tfrac{4}{3}\vec{PR}$$
$$= (1,3,7) + \tfrac{4}{3}(3,-3,-13)$$
$$= (5,-1,-\tfrac{31}{3})$$

$$\boxed{\therefore\ Q = (5,-1,-\tfrac{31}{3})}$$

[3]

Question 2: Determine all values of t, if any, for which $\mathbf{u} = (4, -1)$ is parallel to $\mathbf{v} = (1, t^2)$.

$$4 = k \cdot 1 \Rightarrow k = 4$$
$$-1 = kt^2 \Rightarrow \therefore -1 = 4t^2$$
$$t^2 = -\tfrac{1}{4}$$

$\boxed{\therefore \text{ no solutions.}}$

[3]

Question 3: Determine if the point $(-8, 8, 3, -1, 7)$ lies on the plane through the origin that is parallel to $\mathbf{u} = (2, 1, 0, 1, -1)$ and $\mathbf{v} = (-2, 3, 1, 0, 2)$.

Does $a\vec{u} + b\vec{v} = (-8,8,3,-1,7)$ have a solution?

$$2a - 2b = -8$$
$$a + 3b = 8$$
$$\left.\begin{array}{l} 0a + b = 3 \\ 1a + 0b = -1 \end{array}\right\} \Rightarrow a = -1,\ b = 3$$
$$-1a + 2b = 7$$

, and these values satisfy the other equations, so $\boxed{\text{yes}}$, $(-8,8,3,-1,7)$ does lie on the plane.

[4]

Question 4: Find a unit vector parallel to the yz-plane that is orthogonal to $\mathbf{v} = (2, 5, -3)$.

Let $\vec{u} = (0, a, b)$ be a vector $\parallel$ to yz-plane.

$\vec{u} \cdot \vec{v} = 0 \Rightarrow 5a - 3b = 0$: take $a = 3$, $b = 5$.

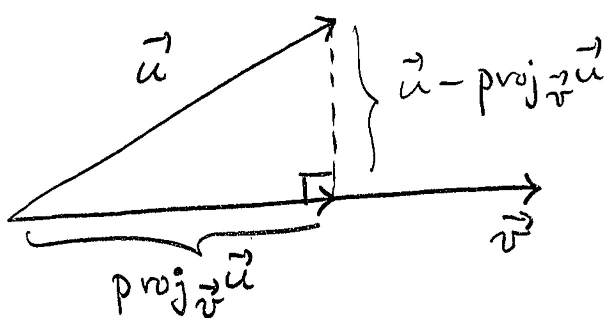
$\therefore \vec{u} = (0, 3, 5)$.

$\therefore$ a unit vector $\parallel$ to $\vec{u}$ is then

$$\frac{\vec{u}}{\|\vec{u}\|} = \frac{(0,3,5)}{\sqrt{0^2+3^2+5^2}} = \boxed{\left(0, \frac{3}{\sqrt{34}}, \frac{5}{\sqrt{34}}\right).}$$

[3]

Question 5: Simplify $(\mathbf{u} - \text{proj}_{\mathbf{v}}\mathbf{u}) \cdot \text{proj}_{\mathbf{v}}\mathbf{u}$ (a sketch may help.)



$\therefore (\vec{u} - \text{proj}_{\vec{v}}\vec{u}) \cdot \text{proj}_{\vec{v}}\vec{u}$

$= \boxed{0}$.

[3]

Question 6: Determine the area of the triangle in $\mathbb{R}^3$ with vertices $P_1(4, 1, -1)$, $P_2(-2, 0, 1)$ and $P_3(-1, 0, 2)$.

$$\text{Area} = \frac{1}{2} \left\| (\vec{P_1P_2}) \times (\vec{P_1P_3}) \right\|$$

$$= \frac{1}{2} \left\| \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -6 & -1 & 2 \\ -5 & -1 & 3 \end{vmatrix} \right\|$$

$$= \frac{1}{2} \|(-1, 8, 1)\|$$

$$= \frac{1}{2}\sqrt{1+64+1}$$

$$= \frac{\sqrt{66}}{2}$$

$$= \boxed{\sqrt{\frac{33}{2}}}$$

[4]

Question 7: Find an equation of the plane containing the lines $\mathbf{x} = (5, 0, -2) + t(-1, 1, 3)$ and $\mathbf{x} = (4, 1, 1) + t(2, 1, -2)$. State your answer in the form $ax + by + cz = d$.

Vectors $\vec{u} = (-1,1,3)$ and $\vec{v} = (2,1,-2)$ are $\parallel$ to plane.

$\therefore$ normal to plane is $\vec{n} = \vec{u} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 1 & 3 \\ 2 & 1 & -2 \end{vmatrix} = (-5,4,-3)$

Point $(5,0,-2)$ is in plane.

$\therefore$ Equation is $\big((x,y,z) - (5,0,-2)\big) \cdot (-5,4,-3) = 0$

$$-5(x-5) + 4(y-0) - 3(z+2) = 0$$

$$-5x + 4y - 3z = -19$$

$$\boxed{5x - 4y + 3z = 19}$$

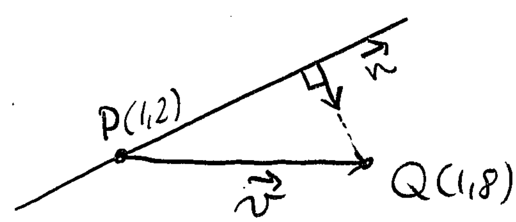
[5]

Question 8: Find the distance from the point $Q(1, 8)$ to the line $3x + y = 5$.

$P(1,2)$ is a point on the line.

$\vec{n} = \dfrac{(3,1)}{\sqrt{10}}$ is a unit normal to line

Let $\vec{v} = \vec{PQ} = (0,6)$



distance is $\|\text{proj}_{\vec{n}} \vec{v}\| = |\vec{v} \cdot \vec{n}|$

$$= \left|(0,6) \cdot \frac{(3,1)}{\sqrt{10}}\right|$$

$$= \boxed{\frac{3\sqrt{2}}{\sqrt{5}}}$$

[5]

Question 9: Find an equation of the plane that contains the point $(7, 0, -2)$ and that is parallel to the plane $\mathbf{x} = (1, 2, 3) + r(-1, 1, 4) + t(0, 3, -2)$. State your answer in the form $ax + by + cz = d$.

normal is $\vec{n} = (-1,1,4) \times (0,3,-2)$

$$= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -1 & 1 & 4 \\ 0 & 3 & -2 \end{vmatrix}$$

$$= (-14, -2, -3)$$

∴ Equation is $\left((x,y,z) - (7,0,-2)\right) \cdot (-14,-2,-3) = 0$

$$-14(x-7) - 2y - 3(z+2) = 0$$

$$-14x - 2y - 3z = -92$$

$$\boxed{14x + 2y + 3z = 92}$$

[5]

Question 10: Determine whether the points $P_1(4, 1, -1)$, $P_2(-2, 0, 1)$, $P_3(1, 1, 1)$ and $P_4(-1, 0, 2)$ all lie in the same plane.

let $\vec{u} = \overrightarrow{P_1P_2} = (-6,-1,2)$

$\vec{v} = \overrightarrow{P_1P_3} = (-3, 0, 2)$

$\vec{w} = \overrightarrow{P_1P_4} = (-5,-1,3)$

$$\vec{u} \cdot (\vec{v} \times \vec{w}) = \begin{vmatrix} -6 & -1 & 2 \\ -3 & 0 & 2 \\ -5 & -1 & 3 \end{vmatrix}$$

$$= -6(0+2) + 1(-9+10) + 2(3-0)$$

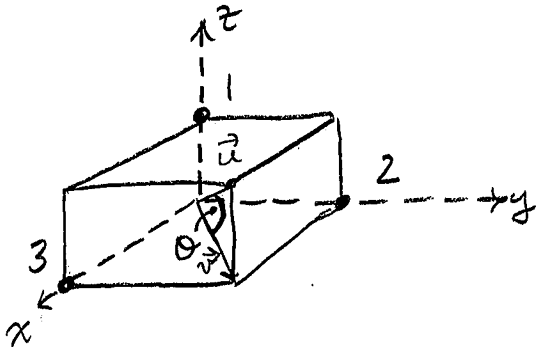
$$= -5$$

$$\neq 0$$

∴ Points do <u>not</u> lie on same plane.

[5]

Question 11: A box has length 3, width 2 and height 1. Consider the line segment joining the two corners of the box that are furthest apart. What angle does this line make with the bottom of the box? State your answer in degrees rounded to one decimal place.



Angle θ is angle between $\vec{u} = (3,2,1)$ and $\vec{v} = (3,2,0)$

$$\vec{u} \cdot \vec{v} = \|\vec{u}\| \, \|\vec{v}\| \cos\theta$$

$$\therefore \theta = \arccos\left[\frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\,\|\vec{v}\|}\right]$$

$$= \arccos\left[\frac{(3,2,1)\cdot(3,2,0)}{\sqrt{9+4+1}\,\sqrt{9+4}}\right]$$

$$= \arccos\left(\sqrt{\frac{13}{14}}\right)$$

$$\approx \boxed{15.5^\circ}$$

[5]

Question 12: Let $\mathbf{u} = (c, 1, c)$, $\mathbf{v} = (1, c, 1)$, $\mathbf{w} = (c, 1, 1)$. Determine all values of c for which $\operatorname{span}\{\mathbf{u}, \mathbf{v}, \mathbf{w}\} = \mathbb{R}^3$.

$$\det\left[\vec{u} \,|\, \vec{v} \,|\, \vec{w}\right] = \begin{vmatrix} c & 1 & c \\ 1 & c & 1 \\ c & 1 & 1 \end{vmatrix}$$

$$= c(c-1) - 1(1-c) + c(1-c^2)$$

$$= c(c-1) + (c-1) + c(1-c)(1+c)$$

$$= c(c-1) + (c-1) - c(c-1)(1+c)$$

$$= (c-1)\left[c+1-c(1+c)\right]$$

$$= (c-1)\left[1-c^2\right]$$

$$\neq \text{ for } c \neq 1, -1$$

$\therefore \operatorname{span}\{\vec{u}, \vec{v}, \vec{w}\} = \mathbb{R}^3$ for $c \neq 1, -1$

[5]