**Question 1:** Let $\mathbf{A} = \begin{bmatrix} 7 & 2 & 1 \\ 0 & 3 & -1 \\ -3 & 4 & -2 \end{bmatrix}$.

(a) Determine $\mathbf{A}^{-1}$ or show that it does not exist. Use any method you wish.

$$\left[\begin{array}{ccc|ccc} 7 & 2 & 1 & 1 & 0 & 0 \\ 0 & 3 & -1 & 0 & 1 & 0 \\ -3 & 4 & -2 & 0 & 0 & 1 \end{array}\right]$$

$R_1 = 2r_3 + r_1$:

$$\left[\begin{array}{ccc|ccc} 1 & 10 & -3 & 1 & 0 & 2 \\ 0 & 3 & -1 & 0 & 1 & 0 \\ -3 & 4 & -2 & 0 & 0 & 1 \end{array}\right]$$

$R_3 = 3r_1 + r_3$:

$$\left[\begin{array}{ccc|ccc} 1 & 10 & -3 & 1 & 0 & 2 \\ 0 & 3 & -1 & 0 & 1 & 0 \\ 0 & 34 & -11 & 3 & 0 & 7 \end{array}\right]$$

$r_2 \leftrightarrow r_3$:

$$\left[\begin{array}{ccc|ccc} 1 & 10 & -3 & 1 & 0 & 2 \\ 0 & 34 & -11 & 3 & 0 & 7 \\ 0 & 3 & -1 & 0 & 1 & 0 \end{array}\right]$$

$R_2 = -11r_3 + r_2$:

$$\left[\begin{array}{ccc|ccc} 1 & 10 & -3 & 1 & 0 & 2 \\ 0 & 1 & 0 & 3 & -11 & 7 \\ 0 & 3 & -1 & 0 & 1 & 0 \end{array}\right]$$

$R_1 = -10r_2 + r_1$:
$R_3 = -3r_2 + r_3$:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -3 & -29 & +110 & -68 \\ 0 & 1 & 0 & 3 & -11 & 7 \\ 0 & 0 & -1 & -9 & 34 & -21 \end{array}\right]$$

$R_3 = (-1)r_3$:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -3 & -29 & 110 & -68 \\ 0 & 1 & 0 & 3 & -11 & 7 \\ 0 & 0 & 1 & 9 & -34 & 21 \end{array}\right]$$

$R_1 = 3r_3 + r_1$:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 8 & -5 \\ 0 & 1 & 0 & 3 & -11 & 7 \\ 0 & 0 & 1 & 9 & -34 & 21 \end{array}\right]$$

$\therefore A^{-1} = \begin{bmatrix} -2 & 8 & -5 \\ 3 & -11 & 7 \\ 9 & -34 & 21 \end{bmatrix}$

[7]

(b) Based on your result from part (a), how many solutions $(x_1, x_2, x_3)$ does the system

$$\mathbf{A} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

have? Explain.

One: $\det(A) \neq 0$, so $A^{-1}$ exists, so

$$A^{-1}A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = A^{-1} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \vec{0}$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \vec{0}.$$

[3]

**Question 2:** Determine **A** if $(3\mathbf{I} - 2\mathbf{A}^\top)^{-1} = \begin{bmatrix} -1 & 2 \\ -3 & 5 \end{bmatrix}$.

$$\therefore \quad 3I - 2A^T = \begin{bmatrix} -1 & 2 \\ -3 & 5 \end{bmatrix}^{-1} = \frac{1}{1}\begin{bmatrix} 5 & -2 \\ 3 & -1 \end{bmatrix}$$

$$\therefore \quad 2A^T = 3I - \begin{bmatrix} 5 & -2 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ -3 & 4 \end{bmatrix}$$

$$\therefore \quad A^T = \frac{1}{2}\begin{bmatrix} -2 & 2 \\ -3 & 4 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -\frac{3}{2} & 2 \end{bmatrix}$$

$$\boxed{\therefore A = \begin{bmatrix} -1 & -3/2 \\ 1 & 2 \end{bmatrix}}$$

**[3]**

**Question 3:** Let $\mathbf{B} = \begin{bmatrix} 8 & 1 & 5 \\ 2 & -7 & -1 \\ 3 & 4 & 1 \end{bmatrix}$ and $\mathbf{F} = \begin{bmatrix} 8 & 1 & 5 \\ 8 & 1 & 1 \\ 3 & 4 & 1 \end{bmatrix}$. Find a matrix **E** so that $\mathbf{EF} = \mathbf{B}$.

$$F \xrightarrow{R_2 = (-2)r_3 + r_2} B, \quad \text{so} \quad E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

**[3]**

**Question 4:** Show that

$$\mathbf{A} = \begin{bmatrix} 0 & a & 0 & 0 & 0 \\ b & 0 & c & 0 & 0 \\ 0 & d & 0 & e & 0 \\ 0 & 0 & f & 0 & g \\ 0 & 0 & 0 & h & 0 \end{bmatrix}$$

is not invertible for any values of $a, b, c, d, e, f, g, h$

using row 1

$$\det(A) = -a \begin{vmatrix} b & c & 0 & 0 \\ 0 & 0 & e & 0 \\ 0 & f & 0 & g \\ 0 & 0 & h & 0 \end{vmatrix}$$

using col 1

$$= -ab \begin{vmatrix} 0 & e & 0 \\ f & 0 & g \\ 0 & h & 0 \end{vmatrix}$$

using col 1

$$= -abf \underbrace{\begin{vmatrix} e & 0 \\ h & 0 \end{vmatrix}}$$

$$= -abf \cdot (0)$$

$$= 0$$

$\therefore$ $A^{-1}$ does not exist.

**[4]**

**Question 5:** How many $2 \times 2$ diagonal matrices satisfy $\mathbf{A}^2 - 3\mathbf{A} - 4\mathbf{I} = \mathbf{0}$ ?

Let $a_{ii} = x$ where $i = 1$ or $2$.

Then $x^2 - 3x - 4 = 0$

$(x-4)(x+1) = 0$

$x = 4,\ x = -1$

$\therefore$ each diagonal element is $4$ or $-1$,

so $A = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix}$ or $\begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix}$.

$\therefore$ 4 possibilities for $A$.

**[3]**

---

**Question 6:** Compute

$$A = \begin{vmatrix} 2 & 0 & 1 & 3 \\ 1 & 1 & 3 & 2 \\ 1 & 0 & -1 & 2 \\ 3 & -1 & 2 & 4 \end{vmatrix}$$

$r_1 \leftrightarrow r_2$:
$$-\det(A) = \begin{vmatrix} 1 & 1 & 3 & 2 \\ 2 & 0 & 1 & 3 \\ 1 & 0 & -1 & 2 \\ 3 & -1 & 2 & 4 \end{vmatrix}$$

$R_2 = (-2)r_1 + r_2$;
$R_3 = (-1)r_1 + r_3$;
$R_4 = (-3)r_1 + r_4$:
$$-\det(A) = \begin{vmatrix} 1 & 1 & 3 & 2 \\ 0 & -2 & -5 & -1 \\ 0 & -1 & -4 & 0 \\ 0 & -4 & -7 & -2 \end{vmatrix}$$

$r_2 \leftrightarrow r_3$:
$$\det(A) = \begin{vmatrix} 1 & 1 & 3 & 2 \\ 0 & -1 & -4 & 0 \\ 0 & -2 & -5 & -1 \\ 0 & -4 & -7 & -2 \end{vmatrix}$$

$R_3 = (-2)r_2 + r_3$;
$R_4 = (-4)r_2 + r_4$:
$$\det(A) = \begin{vmatrix} 1 & 1 & 3 & 2 \\ 0 & -1 & -4 & 0 \\ 0 & 0 & 3 & -1 \\ 0 & 0 & 9 & -2 \end{vmatrix}$$

$R_4 = (-3)r_3 + r_4$:
$$\det(A) = \begin{vmatrix} 1 & 1 & 3 & 2 \\ 0 & -1 & -4 & 0 \\ 0 & 0 & 3 & -1 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

$$= \boxed{-3}$$

**[7]**

**Question 7:** Find the deteminant of the $n \times n$ matrix

$$A = \begin{bmatrix} (1-n) & 1 & 1 & \cdots & 1 \\ 1 & (1-n) & 1 & \cdots & 1 \\ 1 & 1 & (1-n) & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & (1-n) \end{bmatrix}$$

Do $R_1 = (1) r_j + r_1$ for $j = 2, 3, 4, \ldots, n$.

This produces a row of zeros in row 1.

$\therefore \det(A) = \boxed{0}$

[4]

---

**Question 8:** Compute $\mathbf{A}^{-1}$ where $\mathbf{A} = \begin{bmatrix} 2 & 0 & 0 \\ 8 & 1 & 0 \\ -5 & 3 & 6 \end{bmatrix}$. (Using adjoints here is likely easier.)

$\det(A) = (2)(1)(6) = 12$

$$\operatorname{adj}(A) = \begin{bmatrix} 6 & -48 & 29 \\ 0 & 12 & -6 \\ 0 & 0 & 2 \end{bmatrix}^T = \begin{bmatrix} 6 & 0 & 0 \\ -48 & 12 & 0 \\ 29 & -6 & 2 \end{bmatrix}.$$

$$\therefore A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A) = \frac{1}{12} \begin{bmatrix} 6 & 0 & 0 \\ -48 & 12 & 0 \\ 29 & -6 & 2 \end{bmatrix} = \boxed{\begin{bmatrix} 1/2 & 0 & 0 \\ -4 & 1 & 0 \\ \frac{29}{12} & -\frac{1}{2} & \frac{1}{6} \end{bmatrix}}.$$

[3]

---

**Question 9:** If $\mathbf{A}$ and $\mathbf{B}$ are both $3 \times 3$ with $\det(\mathbf{A}) = 4$ and $\det(\mathbf{B}) = -3$, determine the value of $\det[(2\mathbf{A})^{-1}(5\mathbf{B})^T]$.

$$= \det\left((2A)^{-1}\right) \det\left((5B)^T\right)$$

$$= \frac{1}{\det(2A)} \cdot \det(5B)$$

$$= \frac{1}{2^3 \det(A)} \cdot 5^3 \det(B)$$

$$= \frac{1}{(8)(4)} \cdot (125)(-3)$$

$$= \boxed{-\frac{375}{32}}$$

[3]

**Question 10:** Use Cramer's rule to solve for $z$:

$$4x + y + z = 3$$
$$3x + 7y - z = 0$$
$$7x + 3y - 5z = 0$$
$$x + z + 2w = 1$$

$$A = \begin{array}{c} \begin{matrix} x & y & z & w \end{matrix} \\ \begin{bmatrix} 4 & 1 & 1 & 0 \\ 3 & 7 & -1 & 0 \\ 7 & 3 & -5 & 0 \\ 1 & 0 & 1 & 2 \end{bmatrix} \end{array} ; \quad \det(A) = 2\left[4(-32) - (1)(-8) + (1)(-40)\right]$$

$$= -320$$

$$A_z = \begin{bmatrix} 4 & 1 & 3 & 0 \\ 3 & 7 & 0 & 0 \\ 7 & 3 & 0 & 0 \\ 1 & 0 & 1 & 2 \end{bmatrix} ; \quad \det(A_z) = (2)(3)\left[9 - 49\right] = -240$$

$$\therefore z = \frac{\det(A_z)}{\det(A)} = \frac{-240}{-320} = \boxed{\frac{3}{4}}$$

**[6]**

---

**Question 11:** Let $\mathbf{A} = \begin{bmatrix} 1 & (k-1) & 7 \\ 2 & (k-3) & 4 \\ 5 & (k+1) & 0 \end{bmatrix}$. Find all value of $k$ for which $\mathbf{A}$ is invertible.

$$A = \begin{bmatrix} 1 & (k-1) & 7 \\ 2 & (k-3) & 4 \\ 5 & (k+1) & 0 \end{bmatrix} \begin{matrix} 1 & k-1 \\ 2 & k-3 \\ 5 & k+1 \end{matrix}$$

$$\det(A) = 20(k-1) + 14(k+1) - 35(k-3) - 4(k+1)$$

$$= -5k + 72$$

$$\det(A) = 0 \quad \text{for} \quad k = \frac{72}{5}$$

$$\therefore A \text{ invertible for } k \neq \frac{72}{5}.$$

**[4]**