Question 1: Suppose you are using the trapezoid rule on 4 subintervals to approximate $\int_1^5 x\ln(x)\,dx$. Determine an error bound $|E_{T_4}|$ on the resulting approximation.

Recall: the error in using the trapezoid rule to approximate $\int_a^b f(x)\,dx$ using n subintervals is at most $\frac{K(b-a)^3}{12n^2}$ where $|f''(x)| \le K$ on $[a, b]$.

$f(x) = x\ln(x)$

$f'(x) = \ln(x) + x\left(\frac{1}{x}\right) = \ln(x) + 1$

$f''(x) = \frac{1}{x}$

$\therefore\ |f''(x)| \le \frac{1}{1} = 1$ on $[1, 5]$.

$\therefore\ K = 1.$

$\therefore\ |E_{T_4}| \le \frac{K(b-a)^3}{12n^2} = \frac{1\cdot(5-1)^3}{12\cdot 4^2} = \boxed{\frac{1}{3}}$

[5]

Question 2: Evaluate the improper integral $\int_1^\infty \frac{\ln(x)}{x^3}\,dx$ making proper use of any required limits.

For $\int \frac{\ln(x)}{x^3}dx$: by parts: let $u = \ln(x)$ $dv = x^{-3}\,dx$

$du = \frac{1}{x}dx$ $v = -\frac{x^{-2}}{2}$

$\therefore \int \frac{\ln(x)}{x^3}dx = \ln(x)\left(-\frac{x^{-2}}{2}\right) - \int -\frac{x^{-2}}{2}\,\frac{1}{x}\,dx$

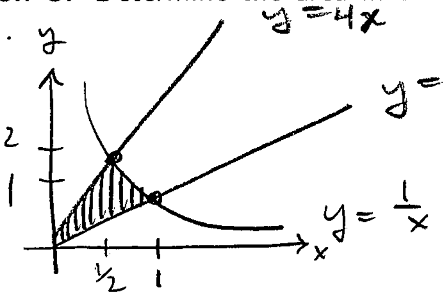
$= \frac{-\ln(x)}{2x^2} + \frac{1}{2}\int x^{-3}\,dx$

$= \frac{-\ln(x)}{2x^2} - \frac{1}{4}\,\frac{1}{x^2} + C$

$\therefore \int_1^\infty \frac{\ln(x)}{x^3}dx = \lim_{b\to\infty}\int_1^b \frac{\ln(x)}{x^3} = \lim_{b\to\infty}\left[\frac{-\ln(x)}{2x^2} - \frac{1}{4}\,\frac{1}{x^2}\right]_1^b$

$= \lim_{b\to\infty}\left[\frac{-\ln(b)}{2b^2} - \frac{1}{4}\,\frac{1}{b^2} - 0 + \frac{1}{4}\right] \stackrel{H}{=} \lim_{b\to\infty}\left[\frac{-1}{4b^2} - \frac{1}{4b^2} + \frac{1}{4}\right] = \boxed{\frac{1}{4}}$ [5]

Question 3: Determine the area in the first quadrant that is bounded by the curves $y = 1/x$, $y = x$ and $y = 4x$.



$x = \frac{1}{x} \Rightarrow x = 1, y = 1$

$4x = \frac{1}{x} \Rightarrow x = \frac{1}{2}, y = 2$

$$\therefore A = \int_0^{1/2} 4x - x\,dx + \int_{1/2}^{1} \frac{1}{x} - x\,dx$$

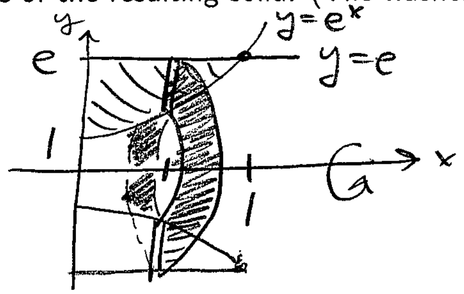
$$= \frac{3}{2}\left[x^2\right]_0^{1/2} + \left[\ln|x|\right]_{1/2}^{1} - \left[\frac{x^2}{2}\right]_{1/2}^{1}$$

$$= \frac{3}{2}\left(\frac{1}{2}\right)^2 - 0 + 0 - \ln|\tfrac{1}{2}| - \left(\frac{1}{2} - \frac{1}{2}\left(\frac{1}{2}\right)^2\right)$$

$$= \cancel{\frac{3}{8}} - \ln(\tfrac{1}{2}) - \cancel{\frac{3}{8}} = \boxed{-\ln(\tfrac{1}{2})} \text{ or } \boxed{\ln(2)}$$

[5]

Question 4: The region bounded by $y = e^x$, $y = e$ and the y-axis is rotated about the x-axis. Determine the volume of the resulting solid. (The washer method would be best here.)



$$V = \int_0^1 \pi\left[e^2 - (e^x)^2\right] dx$$

$$= \pi\int_0^1 e^2 - e^{2x}\,dx$$

$$= \pi\left[e^2 x - \frac{e^{2x}}{2}\right]_0^1$$

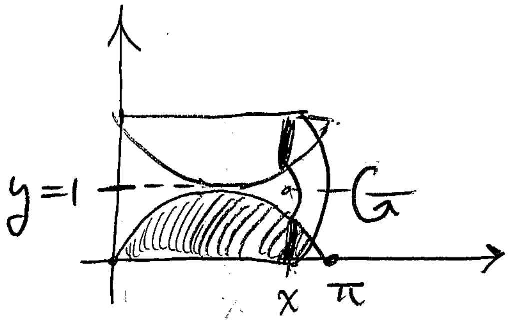
$$= \pi\left[\left(e^2 - \frac{e^2}{2}\right) - \left(0 - \frac{1}{2}\right)\right]$$

$$= \boxed{\pi\left(\frac{e^2+1}{2}\right)}$$

[5]

Question 5: For this question let R be the region between the curve $y = \sin(x)$ and the x-axis over $0 \le x \le \pi$. State integrals giving the volume of each of the following solids. Use whichever method (disks, washers or cylindrical shells) you think is best. DO NOT EVALUATE THE INTEGRALS, JUST SET THEM UP.

(a) Rotate R about the horizontal line $y = 1$.

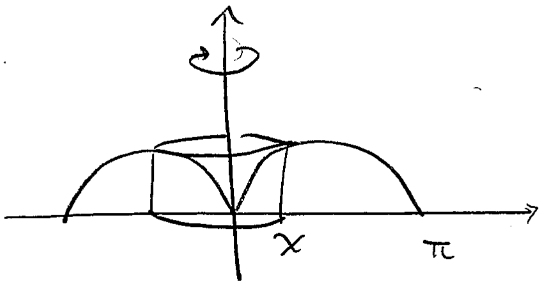


By washers:

$$V = \int_0^{\pi} \pi \left[1^2 - (1-\sin(x))^2\right] dx$$

[3]

(b) Rotate R about the y-axis.

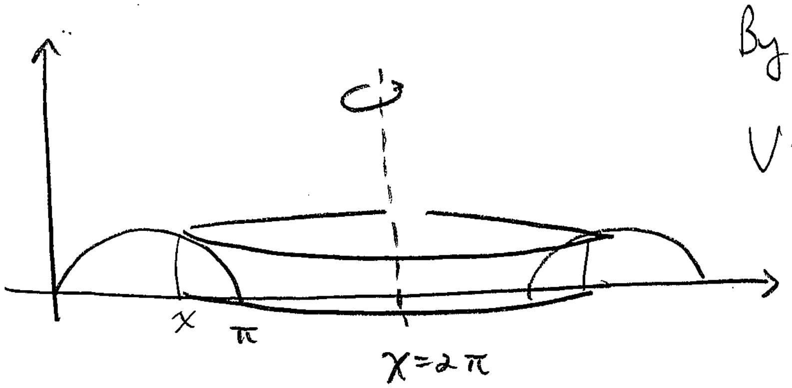


By cylindrical shells:

$$V = \int_0^{\pi} 2\pi x \sin(x)\, dx$$

[3]

(c) Rotate R about the vertical line $x = 2\pi$



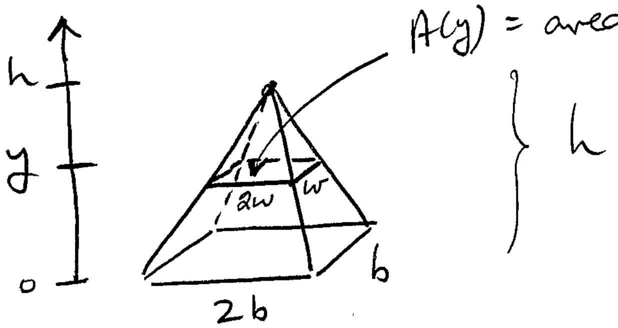
By cylindrical shells:

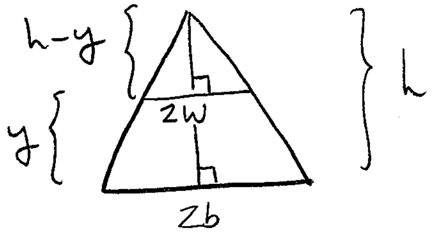
$$V = \int_0^{\pi} 2\pi (2\pi - x) \sin(x)\, dx$$

[4]

Question 6: Determine the volume of the pyramid of height h and rectangular base of side lengths b and $2b$.

$A(y)$ = area of cross-section at y





By similar Δ's : $\dfrac{h-y}{w} = \dfrac{h}{b}$

$$\therefore \quad w = \frac{b(h-y)}{h} = b\left(1-\frac{y}{h}\right)$$

$$\therefore \quad A(y) = (2w)(w) = 2b^2\left(1-\frac{y}{h}\right)^2$$

$$\therefore \quad V = \int_0^h 2b^2\left(1-\frac{y}{h}\right)^2 dy$$

$$= 2b^2 \int_0^h \left(1 - 2\frac{y}{h} + \frac{1}{h^2}y^2\right) dy$$

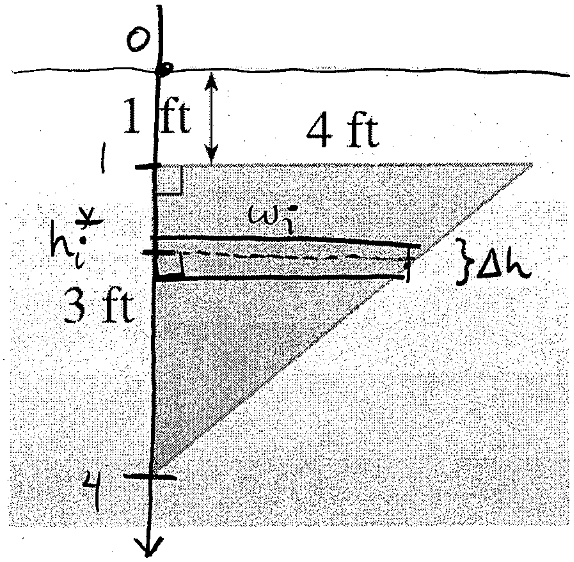
$$= 2b^2 \left[y - \frac{\cancel{2}}{h}\frac{y^2}{\cancel{2}} + \frac{1}{h^2}\frac{y^3}{3} \right]_0^h$$

$$= 2b^2\left[\cancel{h} - \frac{h\cancel{^2}}{\cancel{h}} + \frac{1}{h^2}\frac{h^3}{3}\right]$$

$$= \boxed{\frac{2b^2h}{3}}$$

[10]

Question 7: A triangular plate is submerged in water as indicated in the figure below. Determine the hydrostatic force against one side of the plate. Recall that the pressure at depth h below the surface is given by $P(h) = \delta h$ where $\delta = 62.5$ lb/ft^3 is the weight density for water. You may leave the constant δ in your final answer.



By similar Δ's:

$$\frac{4-h_i^*}{w_i} = \frac{3}{4}$$

$$\Rightarrow w_i = \frac{4}{3}(4-h_i^*)$$

$\therefore$ Area of strip at h_i^* is

$$A_i = w_i\,\Delta h = \frac{4}{3}(4-h_i^*)\Delta h$$

Pressure at h_i^* is

$$P_i = \delta h_i^*$$

$\therefore$ Force on strip at h_i^* is

$$F_i \approx A_i P_i = \frac{4}{3}\delta(4-h_i^*)\,h_i^*\Delta h$$

$\therefore$ Total force is

$$F \approx \sum_{i=1}^{n} \frac{4}{3}\delta(4-h_i^*)\,h_i^*\Delta h.$$

Letting $n \to \infty$:

$$F = \int_1^4 \frac{4}{3}\delta\,(4-h)h\,dh$$

$$= \frac{4}{3}\delta \int_1^4 4h - h^2\,dh$$

$$= \frac{4}{3}\delta\left[\frac{4}{2}h^2 - \frac{h^3}{3}\right]_1^4$$

$$= \frac{4}{3}\delta\left[2\cdot 16 - \frac{4^3}{3} - 2 + \frac{1}{3}\right]$$

$$= \frac{4}{3}\delta\left[30 - \frac{63}{3}\right]$$

$$= \frac{4}{3}\delta\left[30 - 21\right]$$

$$= \boxed{12\delta \text{ lbs.}}$$

[10]