Question 1: Use the definition of the definite integral in the form

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x$$

to evaluate

$$\int_1^2 (x^2+1)\,dx$$

Carefully set up the Riemann sum and clearly show the steps of your simplification.

$$[a,b] = [1,2]$$

$$\Delta x = \frac{b-a}{n} = \frac{2-1}{n} = \frac{1}{n}$$

$$x_i = a + i\Delta x = 1 + \frac{i}{n}$$

$$f(x) = x^2+1 \Rightarrow f(x_i) = \left(1+\frac{i}{n}\right)^2 + 1 = \frac{i^2}{n} + 2\frac{i}{n} + 2$$

$$\int_1^2 (x^2+1)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x$$

$$= \lim_{n\to\infty}\sum_{i=1}^{n}\left[\frac{i^2}{n^2} + 2\frac{i}{n} + 2\right]\left(\frac{1}{n}\right)$$

$$= \lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{i^2}{n^3} + \frac{2i}{n^2} + \frac{2}{n}\right)$$

$$= \lim_{n\to\infty}\left[\frac{1}{n^3}\left(\sum_{i=1}^{n} i^2\right) + \frac{2}{n^2}\left(\sum_{i=1}^{n} i\right) + \frac{2}{n}\left(\sum_{i=1}^{n} 1\right)\right]$$

$$= \lim_{n\to\infty}\left[\frac{1}{6}\frac{\cancel{n}}{\cancel{n}}\underbrace{\frac{(n+1)}{n}}_{\to 1}\underbrace{\frac{(2n+1)}{n}}_{\to 2} + \frac{\cancel{2}}{\cancel{2}}\frac{\cancel{n}}{\cancel{n}}\underbrace{\frac{(n+1)}{n}}_{\to 1} + \frac{2}{\cancel{n}}\cdot\cancel{n}\right]$$

$$= \left(\frac{1}{6}\right)(1)(2) + 1 + 2$$

$$= \boxed{\frac{10}{3}}$$

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Question 2: Water flows from a tank at a rate of $r(t) = 200 - 2t^2$ litres per minutes, where $0 \le t \le 10$. How much water flows from the tank over the first 10 minutes? State units with your answer.

$$V = \int_0^{10} 200 - 2t^2 \, dt$$

$$= \left[200t - \frac{2t^3}{3} \right]_0^{10}$$

$$= \left(2000 - \frac{2000}{3}\right) - (0 - 0)$$

$$= \boxed{\frac{4000}{3} \text{ L}}$$

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Question 3: Find the number c in the interval $[2, 5]$ with the property that for $f(x) = (x-3)^2$, $f_{\text{ave}} = f(c)$.

$$f_{ave} = \frac{1}{5-2} \int_2^5 (x-3)^2 dx = \frac{1}{3} \left[\frac{(x-3)^3}{3} \right]_2^5 = \frac{1}{9} \left[8 - (-1) \right] = 1$$

Need c so that $(c-3)^2 = 1$

$$c - 3 = \pm 1$$

$$\boxed{c = 4} \text{ or } \boxed{c = 2}$$

[3]

Question 4: Suppose f is a continuous function with the property

$$\int_0^x f(t)\, dt = xe^{2x} + \int_0^x e^{-t} f(t)\, dt \, .$$

Find a formula for $f(x)$.

$$\frac{d}{dx} \int_0^x f(t) dt = \frac{d}{dx} \left[xe^{2x} + \int_0^x e^{-t} f(t)\, dt \right]$$

$$f(x) = e^{2x} + 2xe^{2x} + e^{-x} f(x)$$

$$\Rightarrow \quad f(x) \left[1 - e^{-x} \right] = e^{2x} + 2xe^{2x}$$

$$\boxed{f(x) = \frac{e^{2x}(1 + 2x)}{1 - e^{-x}}}.$$

[4]

Question 5: Evaluate the following integrals:

(a) $\displaystyle\int \left(\frac{\sqrt{x}}{2} - e^x + 1\right)\, dx$

$$= \int \tfrac{1}{2} x^{1/2} - e^x + 1 \; dx$$

$$= \tfrac{1}{2} \frac{x^{3/2}}{(3/2)} - e^x + x + C \longrightarrow = \boxed{\tfrac{1}{3} x^{3/2} - e^x + x + C}$$

[3]

(b) $\displaystyle\int \left(x^{-1} - \csc(x)\cot(x)\right)\, dx$

$$= \boxed{\ln|x| + \csc(x) + C}$$

[3]

(c) $\displaystyle\int_0^{\pi/4} \left(1 + \cos(x) + 2\sec^2(x)\right)\, dx$

$$= \Big[x + \sin(x) + 2\tan(x)\Big]_0^{\pi/4}$$

$$= \left[\frac{\pi}{4} + \sin\left(\frac{\pi}{4}\right) + 2\tan\left(\frac{\pi}{4}\right)\right] - \left[\cancel{0 + \sin(0) + 2\tan(0)}^{\,0}\right]$$

$$= \frac{\pi}{4} + \frac{1}{\sqrt{2}} + 2 \cdot 1$$

$$= \boxed{\frac{\pi + 2\sqrt{2} + 8}{4}}$$

[4]

Question 6: Evaluate the following integrals:

(a) $\frac{1}{4}\int 4x \sin(2x^2)\,dx = I$

let $u = 2x^2$

$du = 4x\,dx$

$\therefore \quad I = \frac{1}{4}\int \sin(u)\,du = \frac{-1}{4}\cos(u) + C = \boxed{-\frac{1}{4}\cos(2x^2) + C}$

[3]

(b) $2\int \frac{1}{2\sqrt{x}(1+\sqrt{x})^2}\,dx = I$

Let $u = 1 + \sqrt{x}$

$du = \frac{1}{2\sqrt{x}}\,dx$

$\therefore I = 2\int u^{-2}\,du$

$= -2u^{-1} + C$

$= \boxed{\frac{-2}{1+\sqrt{x}} + C}$

[3]

(c) $\int \frac{e^x}{e^x + e^{-x}}\,dx = I$

Let $u = e^x$

$du = e^x\,dx$

$\therefore I = \int \frac{1}{u + u^{-1}}\,du$

$= \frac{1}{2}\int \frac{2u}{u^2+1}\,du$

let $w = u^2 + 1$

$dw = 2u\,du$

$\therefore I = \frac{1}{2}\int \frac{1}{w}\,dw$

$= \frac{1}{2}\ln|w| + C$

$= \frac{1}{2}\ln|u^2+1| + C$

$= \boxed{\frac{1}{2}\ln|e^{2x}+1| + C}$

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Question 7: Evaluate $\int_0^1 (1-3x)e^{2x}\,dx = I$

$$u = 1-3x \quad \Big| \quad dv = e^{2x}dx$$
$$du = -3\,dx \quad \Big| \quad v = \frac{e^{2x}}{2}$$

$$\therefore I = \int_0^1 u\,dv = [uv]_0^1 - \int_0^1 v\,du$$
$$= \left[(1-3x)\frac{e^{2x}}{2}\right]_0^1 - \int_0^1 \frac{e^{2x}}{2}(-3)\,dx$$
$$= -2\frac{e^2}{2} - \frac{1}{2} + \frac{3}{2}\int_0^1 e^{2x}\,dx$$
$$= \frac{-1-2e^2}{2} + \frac{3}{2}\left[\frac{e^{2x}}{2}\right]_0^1$$
$$= \frac{-1-2e^2}{2} + \frac{3}{2}\left(\frac{e^2-1}{2}\right)$$
$$= \frac{-2-4e^2+3e^2-3}{4}$$
$$= \boxed{-\left(\frac{5+e^2}{4}\right)}$$

[5]

Question 8: Evaluate $\int x[\ln(x)]^2\,dx = I$

$$u = [\ln(x)]^2 \qquad dv = x\,dx$$
$$du = \frac{2\ln(x)}{x}dx \qquad v = \frac{x^2}{2}$$

$$I = \int u\,dv$$
$$= uv - \int v\,du$$
$$= [\ln(x)]^2\left(\frac{x^2}{2}\right) - \int \frac{x^{\cancel{2}1}}{\cancel{2}} \cdot \frac{\cancel{2}\ln(x)}{\cancel{x}}dx$$
$$= \frac{x^2[\ln(x)]^2}{2} - \underbrace{\int x\ln(x)\,dx}_{\substack{u=\ln(x) \quad dv = x\,dx \\ du = \frac{1}{x}dx \quad v = \frac{x^2}{2}}}$$

$$= \frac{x^2[\ln(x)]^2}{2} - \left[\frac{x^2}{2}\ln(x) - \int \frac{x^2}{2}\cdot\frac{1}{x}dx\right]$$
$$= \boxed{\frac{x^2[\ln(x)]^2}{2} - \frac{x^2\ln(x)}{2} + \frac{x^2}{4} + C}$$

[5]