Question 1: Determine whether $S = \{4 + 6x + x^2, -1 + 4x + 2x^2, 5 + 2x - x^2\}$ is a basis for P_2.

$$\begin{vmatrix} 4 & -1 & 5 \\ 6 & 4 & 2 \\ 1 & 2 & -1 \end{vmatrix}$$

$$= 4(-4-4) - (-1)(-6-2) + 5(12-4)$$

$$= -32 - 8 + 40$$

$$= 0$$

$\therefore$ S is <u>not</u> a basis for P_2.

[5]

Question 2: Find a basis and state the dimension of the subspace V of P_2 consisting of all polynomials $p(x) = a_0 + a_1x + a_2x^2$ for which $p(0) = 0$.

If $p(0) = 0$ then x is a factor of $p(x)$,
So $p(x)$ has form $a_1x + a_2x^2$.

$p_1(x) = x$, $p_2(x) = x^2$ are two such polynomials.

$\therefore$ span $\{x, x^2\} = V$, and since $\{x, x^2\}$ is a linearly independent set (neither vector is a salar multiple of the other), $\{x, x^2\}$ is a basis.

$\therefore$ V has dimension 2 and basis $\{x, x^2\}$.

[5]

Question 3: Let $\mathbf{u}_1 = (2, -6)$, $\mathbf{u}_2 = (3, 8)$, $\mathbf{w} = (2, 2)$ and $S = \{\mathbf{u}_1, \mathbf{u}_2\}$. Find $(\mathbf{w})_S$. (Recall, $(\mathbf{w})_S$ is the coordinate vector for $\mathbf{w}$ relative to the basis S.)

$(w)_S = (a, b)$ where $a\vec{u}_1 + b\vec{u}_2 = \vec{w}$.

$\therefore$ solve $\begin{bmatrix} 2 & 3 \\ -6 & 8 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} a \\ b \end{bmatrix} = \frac{1}{34} \begin{bmatrix} 8 & -3 \\ 6 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{10}{34} \\ \frac{16}{34} \end{bmatrix} = \begin{bmatrix} \frac{5}{17} \\ \frac{8}{17} \end{bmatrix}$$

$\therefore (w)_S = \left(\frac{5}{17}, \frac{8}{17}\right)$.

[5]

Question 4: Find a basis and state the dimension for the solution space of the linear homogeneous system

$$\begin{aligned} 2x_1 + x_2 + 3x_3 + x_4 &= 0 \\ x_1 \qquad + 5x_3 + x_4 &= 0 \\ x_1 + x_2 - 2x_3 \qquad &= 0 \end{aligned}$$

$$\begin{bmatrix} 2 & 1 & 3 & 1 & 0 \\ 1 & 0 & 5 & 1 & 0 \\ 1 & 1 & -2 & 0 & 0 \end{bmatrix}$$

$r_1 \leftrightarrow r_2$: $\begin{bmatrix} 1 & 0 & 5 & 1 & 0 \\ 2 & 1 & 3 & 1 & 0 \\ 1 & 1 & -2 & 0 & 0 \end{bmatrix}$

$R_2 = (-2)r_1 + r_2$:
$R_3 = (-1)r_1 + r_3$:

$$\begin{bmatrix} 1 & 0 & 5 & 1 & 0 \\ 0 & 1 & -7 & -1 & 0 \\ 0 & 1 & -7 & -1 & 0 \end{bmatrix}$$

$R_3 = (-1)r_2 + r_3$:

$$\begin{bmatrix} 1 & 0 & 5 & 1 & 0 \\ 0 & 1 & -7 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Letting $x_3 = r$, $x_4 = t$:

$x_1 = -5r - t$, $x_2 = 7r + t$

$\therefore$ Solution is
$(-5r-t, 7r+t, r, t) = r(-5, 7, 1, 0) + t(-1, 1, 0, 1)$

$\therefore$ Basis for solution space is
$\{(-5,7,1,0), (-1,1,0,1)\}$; dimension is 2.

[5]

Question 5: Determine if $\mathbf{b} = \begin{bmatrix} -2 \\ 10 \end{bmatrix}$ is in the column space of $\mathbf{A} = \begin{bmatrix} 1 & 3 \\ 4 & -6 \end{bmatrix}$, and if so, express $\mathbf{b}$ as a linear combination of the columns of $\mathbf{A}$.

Solve (if possible) $\begin{bmatrix} 1 & 3 \\ 4 & -6 \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix} = \begin{bmatrix} -2 \\ 10 \end{bmatrix}$.

$$\Rightarrow \begin{bmatrix} p \\ q \end{bmatrix} = \frac{-1}{18} \begin{bmatrix} -6 & -3 \\ -4 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ 10 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$\therefore \vec{b} = (1)\begin{bmatrix} 1 \\ 4 \end{bmatrix} + (-1)\begin{bmatrix} 3 \\ -6 \end{bmatrix}$ so is in the column space of A.

[5]

Question 6: Find a basis for the subspace of R^4 spanned by $\mathbf{v}_1 = (-1, 1, -2, 1)$, $\mathbf{v}_2 = (3, 3, 6, 1)$ and $\mathbf{v}_3 = (-9, -3, -18, 1)$.

$$\begin{bmatrix} -1 & 1 & -2 & 1 \\ 3 & 3 & 6 & 1 \\ -9 & -3 & -18 & 1 \end{bmatrix}$$

$R_1 = (-1) r_1$: $\begin{bmatrix} 1 & -1 & 2 & -1 \\ 3 & 3 & 6 & 1 \\ -9 & -3 & -18 & 1 \end{bmatrix}$

$R_2 = (-3) r_1 + r_2$:
$R_3 = (9) r_1 + r_3$: $\begin{bmatrix} 1 & -1 & 2 & -1 \\ 0 & 6 & 0 & 4 \\ 0 & -12 & 0 & -8 \end{bmatrix}$

$R_2 = \left(\frac{1}{6}\right) r_2$: $\begin{bmatrix} 1 & -1 & 2 & -1 \\ 0 & 1 & 0 & 2/3 \\ 0 & -12 & 0 & -8 \end{bmatrix}$

$R_3 = (12) r_2 + r_3$: $\begin{bmatrix} 1 & -1 & 2 & -1 \\ 0 & 1 & 0 & 2/3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\therefore$ basis is

$\{(1,-1,2,-1), (0,1,0,2/3)\}$.

[5]

Question 7: The matrix $\mathbf{A} = \begin{bmatrix} 1 & 2 & 0 & 0 & 3 \\ 1 & 2 & 1 & 0 & 2 \\ 3 & 6 & 1 & 0 & 8 \\ -1 & -2 & -2 & 0 & -1 \end{bmatrix}$ has RREF $\mathbf{R} = \begin{bmatrix} 1 & 2 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

(a) Give a basis and state the dimension of the row space of $\mathbf{A}$.

basis $\{(1,2,0,0,3), (0,0,1,0,-1)\}$

dimension 2.

[2]

(b) Give a basis and state the dimension of the column space of $\mathbf{A}$.

basis $\{(1,1,3,-1), (0,1,1,-2)\}$

dimension = 2.

[2]

(c) Give a basis and state the dimension of the null space of $\mathbf{A}$. (Equivalently, give a basis and state the dimension of the solution space of $\mathbf{Ax} = \mathbf{0}$.)

Let $x_2 = r$, $x_4 = s$, $x_5 = t$ $\Rightarrow$ $x_3 = t$, $x_1 = -2r-3t$

$\therefore$ solutions are $(-2r-3t, r, t, s, t)$

$= r(-2,1,0,0,0) + s(0,0,0,1,0) + t(-3,0,1,0,1)$

$\therefore$ basis is $\{(-2,1,0,0,0), (0,0,0,1,0), (-3,0,1,0,1)\}$,

dimension is 3.

[3]

(d) Determine rank($\mathbf{A}$) and nullity($\mathbf{A}$) .

rank$(A) = 2$, nullity$(A) = 3$

[1]

(e) Determine rank($\mathbf{A}^T$) and nullity($\mathbf{A}^T$) .

rank$(A^T) = 2$, nullity$(A^T) = 2$.

[1]

Question 8: Let the transformation T_1 represent reflection about the yz-plane:

$$T_1\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = \begin{bmatrix} -a \\ b \\ c \end{bmatrix},$$

and T_2 be the transformation which rotates a vector in R^3 counter-clockwise about the z-axis by an angle θ.

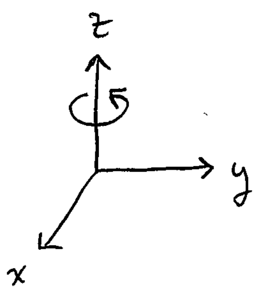
(a) Determine the standard matrix **A** for the transformation T_1.

$$T_1\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}, \quad T_1\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad T_1\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\therefore A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

[3]

(b) Determine the standard matrix **B** for the transformation T_2.



$$T_1\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{bmatrix}, \quad T_2\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{bmatrix}, \quad T_3\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

$$\therefore B = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

[3]

(c) Determine the image of the vector $(1, 2, -1)$ if it is first reflected about the yz-plane and then rotated about the z-axis by $\pi/4$ (or $45°$.)

$$\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{bmatrix}\begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} -3/\sqrt{2} \\ 1/\sqrt{2} \\ -1 \end{bmatrix}.$$

[4]