**Question 1:** Determine an equation of the plane containing the three points $P_1(3,0,2)$, $P_2(4,3,0)$ and $P_3(8,1,-1)$. (You may state your equation of the plane in any form you wish.)

Sol$^n$ 1: $\overrightarrow{P_1P_2} = (1,3,-2)$

$\overrightarrow{P_1P_3} = (5,1,-3)$

$\therefore$ normal to plane is $\vec{n} = \overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3}$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 3 & -2 \\ 5 & 1 & -3 \end{vmatrix}$$

$$= (-7,-7,-14)$$

$$= -7\underbrace{(1,1,2)}$$

$\uparrow$ use $\vec{n}_1 = (1,1,2)$

using $P(3,0,2)$, equation is $(x-3, y, z-2)\cdot(1,1,2) = 0$

or $(x-3) + y + 2(z-2) = 0$

or $x + y + 2z = 7$.

Sol$^n$ 2: using $P_1(3,0,2)$, $\overrightarrow{P_1P_2} = (1,3,-2)$, $\overrightarrow{P_1P_3} = (5,1,-3)$, equation is

$(x,y,z) = (3,0,2) + r(1,3,-2) + t(5,1,-3)$.

**[5]**

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**Question 2:** Determine the distance from the line $\mathbf{r} = (1,3,2) + t(1,2,-1)$ to the plane $y + 2z = 5$.

Set $t=0$ to get the point $P_1(1,3,2)$ on the line.

Set $x = z = 0$ to get the point $P_2(0,5,0)$ on the plane.

Normal to plane is $\vec{n} = (0,1,2)$.

$\therefore$ distance is $d = \| \operatorname{proj}_{\vec{n}} \overrightarrow{P_1P_2} \|$

$$= \| \operatorname{proj}_{(0,1,2)} (-1,2,-2) \|$$

$$= \left| \frac{(-1,2,-2)\cdot(0,1,2)}{\sqrt{0^2+1^2+2^2}} \right|$$

$$= \frac{2}{\sqrt{5}}$$

**[5]**

**Question 3:** Find vector and parametric equations of the plane through the origin that is orthogonal to $\mathbf{v} = (3, 1, -6)$.

Plane is $3(x-0) + 1(y-0) - 6(z-0) = 0$

$\Rightarrow 3x + y - 6z = 0$

$\Rightarrow x = -\frac{1}{3}y + 2z$

Let $y = r$, $z = t$ $\Rightarrow x = \frac{-1}{3}r + 2t$.

$\therefore (x, y, z) = \left(\frac{-1}{3}r + 2t, r, t\right)$

$= r\left(\frac{-1}{3}, 1, 0\right) + t(2, 0, 1)$

$\therefore$ Vector equation is $\vec{x} = (0,0,0) + r\left(\frac{-1}{3}, 1, 0\right) + t(2, 0, 1)$

parametric equations are $x = \frac{-1}{3}r + 2t$

$y = r$

$z = t$.

**[5]**

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**Question 4:** Determine all values of $a$ for which $\mathbf{u} = (-1, 2, 4)$, $\mathbf{v} = (a, 1, 1)$ and $\mathbf{w} = (0, a, 2)$ lie in the same plane.

Need $\vec{u} \cdot (\vec{v} \times \vec{w}) = 0 \Rightarrow \begin{vmatrix} -1 & 2 & 4 \\ a & 1 & 1 \\ 0 & a & 2 \end{vmatrix} = 0$

$\Rightarrow (-1)(2-a) - (2)(2a) + (4)(a^2) = 0$

$\Rightarrow 4a^2 - 3a - 2 = 0$

$\Rightarrow a = \dfrac{-(-3) \pm \sqrt{(-3)^2 - 4(4)(-2)}}{2(4)}$

$= \dfrac{3 \pm \sqrt{41}}{8}$

**[5]**

**Question 5:** Suppose $\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = 5$ . Determine

$$\mathbf{v}\cdot(\mathbf{w}\times\mathbf{u}) - \mathbf{w}\cdot(\mathbf{u}\times\mathbf{v})$$

$$= \begin{vmatrix} v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \\ u_1 & u_2 & u_3 \end{vmatrix} - \begin{vmatrix} w_1 & w_2 & w_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$r_1 \leftrightarrow r_3$ followed by $r_2 \leftrightarrow r_3$

$r_1 \leftrightarrow r_2$ followed by $r_2 \leftrightarrow r_3$

$$= (-1)(-1)\begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} - (-1)(-1)\begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

$$= 0.$$

**[5]**

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**Question 6:** Express (if possible) $\mathbf{b} = (7, 8, 9)$ as a linear combination of $\mathbf{u} = (2, 1, 4)$, $\mathbf{v} = (1, -1, 3)$ and $\mathbf{w} = (3, 2, 5)$.

Solve $k_1\vec{u} + k_2\vec{v} + k_3\vec{w} = \vec{b}$

$$\Rightarrow \begin{bmatrix} 2 & 1 & 3 & 7 \\ 1 & -1 & 2 & 8 \\ 4 & 3 & 5 & 9 \end{bmatrix}$$

$r_1 \leftrightarrow r_2$: $\begin{bmatrix} 1 & -1 & 2 & 8 \\ 2 & 1 & 3 & 7 \\ 4 & 3 & 5 & 9 \end{bmatrix}$

$R_2 = (-2)r_1 + r_2$:
$R_4 = (-4)r_1 + r_4$: $\begin{bmatrix} 1 & -1 & 2 & 8 \\ 0 & 3 & -1 & -9 \\ 0 & 7 & -3 & -23 \end{bmatrix}$

$r_2 \leftrightarrow r_3$ : $\begin{bmatrix} 1 & -1 & 2 & 8 \\ 0 & 7 & -3 & -23 \\ 0 & 3 & -1 & -9 \end{bmatrix}$

$R_2 = (-2)r_3 + r_2$: $\begin{bmatrix} 1 & -1 & 2 & 8 \\ 0 & 1 & -1 & -5 \\ 0 & 3 & -1 & -9 \end{bmatrix}$

$R_3 = (-3)r_2 + r_3$: $\begin{bmatrix} 1 & -1 & 2 & 8 \\ 0 & 1 & -1 & -5 \\ 0 & 0 & 2 & 6 \end{bmatrix}$

$R_3 = \frac{1}{2}r_3$: $\begin{bmatrix} 1 & -1 & 2 & 8 \\ 0 & 1 & -1 & -5 \\ 0 & 0 & 1 & 3 \end{bmatrix}$

$\therefore k_3 = 3$

$k_2 = -5 + k_3 = -2$

$k_1 = 8 + k_2 - 2k_3 = 0$

$\therefore \vec{b} = -2\vec{v} + 3\vec{w}$.

**[5]**

**Question 7:** Determine the values of $b$ for which the solution space of following system is a line through the origin:

$$\begin{aligned} x_1 + x_2 + bx_3 &= 0 \\ x_1 + bx_2 + x_3 &= 0 \\ bx_1 + x_2 + x_3 &= 0 \end{aligned}$$

(notice: $b \neq 1$, otherwise all three equations become $x_1 + x_2 + x_3 = 0$, which is not a line.)

Sol$^n$ 1:

$$\begin{bmatrix} 1 & 1 & b \\ 1 & b & 1 \\ b & 1 & 1 \end{bmatrix}$$

$R_2 = (-1)r_1 + r_2$, $R_3 = (-b)r_1 + r_3$: $\begin{bmatrix} 1 & 1 & b \\ 0 & b-1 & 1-b \\ 0 & 1-b & 1-b^2 \end{bmatrix}$

$R_2 = \left(\frac{1}{b-1}\right) r_2$: $\begin{bmatrix} 1 & 1 & b \\ 0 & 1 & -1 \\ 0 & 1-b & 1-b^2 \end{bmatrix}$

$R_3 = (b-1)r_2 + r_3$: $\begin{bmatrix} 1 & 1 & b \\ 0 & 1 & -1 \\ 0 & 0 & (1-b)+(1-b^2) \end{bmatrix}$

For a line through $(0,0,0)$, need

$(1-b) + (1-b^2) = 0$

$\Rightarrow (1-b)(2+b) = 0$

$\Rightarrow \underbrace{\cancel{b=1}}_{\text{otherwise solution space is a plane}}$, $\boxed{b = -2}$

Sol$^n$ 2:

Since solution space is not $(0,0,0)$ $\Rightarrow \begin{vmatrix} 1 & 1 & b \\ 1 & b & 1 \\ b & 1 & 1 \end{vmatrix} = 0$

$(b-1) - (1-b) + b(1-b^2) = 0$

$(1-b)[-1-1+b(1+b)] = 0$

$(1-b)(b^2+b-2) = 0$

$(1-b)(b-1)(b+2) = 0$

$\therefore b = 1, \ b = -2.$

$b = 1$ corresponds to the plane $x_1 + x_2 + x_3 = 0$.

$\therefore b = -2$ corresponds to either $\mathbb{R}^3$ or a line through $(0,0,0)$. The solution space is not $\mathbb{R}^3$, so $\boxed{b=-2}$ corresponds to a line through $(0,0,0)$.

**[5]**

**Question 8:** Determine whether the functions $f(x) = x$, $g(x) = \sin(2x)$, $h(x) = \cos(2x)$ are linearly independent.

$$W(x) = \begin{vmatrix} f(x) & g(x) & h(x) \\ f'(x) & g'(x) & h'(x) \\ f''(x) & g''(x) & h''(x) \end{vmatrix}$$

$$= \begin{vmatrix} x & \sin(2x) & \cos(2x) \\ 1 & 2\cos(2x) & -2\sin(2x) \\ 0 & -4\sin(2x) & -4\cos(2x) \end{vmatrix}$$

$$= x\left[-8\cos^2(2x) - 8\sin^2(2x)\right] - (1)\cancelto{0}{\left[-4\sin(2x)\cos(2x) + 4\sin(2x)\cos(2x)\right]}$$

$$= -8x \neq 0 \text{ for } x = 1 \text{ (say)}.$$

**[5]**

**Question 9:** Let $S = \{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ where

$$\mathbf{p}_1 = 2 + x + 4x^2, \quad \mathbf{p}_2 = 1 - x + 3x^2, \quad \mathbf{p}_3 = 3 + 2x + 5x^2$$

Determine whether span(S) = $P_2$.

Does $k_1 p_1 + k_2 p_2 + k_3 p_3 = a_0 + a_1 x + a_2 x^2$ have solutions for every choice of $a_0, a_1, a_2$?

Yes iff $\det \begin{bmatrix} 2 & 1 & 3 \\ 1 & -1 & 2 \\ 4 & 3 & 5 \end{bmatrix} \neq 0$

$$\begin{vmatrix} 2 & 1 & 3 \\ 1 & -1 & 2 \\ 4 & 3 & 5 \end{vmatrix} = (2)(-11) - 1(-3) + 3(7) = 2 \neq 0$$

$\therefore$ span$(S) = P_2$.

**[5]**

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**Question 10:** Determine if $S = \{(0,0,2,2), (3,3,0,0), (1,1,0,-1)\}$ is linearly dependent or independent. (Hint: this can be done in several different ways, at least two of which are very short.)

Sol$^n$ 1: Suppose $k_1(0,0,2,2) + k_2(3,3,0,0) + k_3(1,1,0,-1) = (0,0,0,0)$

Using 3$^{rd}$ component: $2k_1 + 0k_2 + 0k_3 = 0 \Rightarrow k_1 = 0$

Using 4$^{th}$ component: $2\underbrace{\cancel{k_1}}_{0} + 0k_2 + (-1)k_3 = 0 \Rightarrow k_3 = 0$

Using 1$^{st}$ component: $0k_1 + 3k_2 + \cancel{k_3}^{0} = 0 \Rightarrow k_2 = 0$.

$\therefore$ S is linearly independent.

Sol$^n$ 2: Again suppose $k_1(0,0,2,2) + k_2(3,3,0,0) + k_3(1,1,0,-1) = (0,0,0,0)$

$$\therefore \left[\begin{array}{ccc|c} 0 & 3 & 1 & 0 \\ 0 & 3 & 1 & 0 \\ 2 & 0 & 0 & 0 \\ 2 & 0 & -1 & 0 \end{array}\right] \xrightarrow{R_1 = (-1)r_2 + r_1} \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 2 & 0 & 0 & 0 \\ 2 & 0 & -1 & 0 \end{array}\right] \Bigg\} \text{Equivalent to } \begin{bmatrix} 0 & 3 & 1 \\ 2 & 0 & 0 \\ 2 & 0 & -1 \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \ (*)$$

Since $\begin{vmatrix} 0 & 3 & 1 \\ 2 & 0 & 0 \\ 2 & 0 & -1 \end{vmatrix} = 6 \neq 0$, only solution to $(*)$ is $k_1 = k_2 = k_3 = 0$.

Sol$^n$ 3: Again suppose $k_1(0,0,2,2) + k_2(3,3,0,0) + k_3(1,1,0,-1) = (0,0,0,0)$

$$\therefore \left[\begin{array}{ccc|c} 0 & 3 & 1 & 0 \\ 0 & 3 & 1 & 0 \\ 2 & 0 & 0 & 0 \\ 2 & 0 & -1 & 0 \end{array}\right] \xrightarrow{\text{Gauss-Jordan}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \therefore k_1 = k_2 = k_3 = 0.$$

**[5]**