Question 1:

(a) Determine all values of x for which $\mathbf{A} = \begin{bmatrix} x & 0 & 0 \\ x^2 & (x^2-1) & 0 \\ x^3 & x^4 & (x^2+1) \end{bmatrix}$ is invertible.

$x \neq 0$, $x^2 - 1 \neq 0$, $\underbrace{x^2+1 \neq 0}_{\text{true for all real } x.}$

$\boxed{\therefore\ x \neq 0,\ x \neq 1,\ x \neq -1}$

[3]

(b) Determine $\mathbf{B}^{-1}$ if $\mathbf{B} = \begin{bmatrix} a^{-1} & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & abc \end{bmatrix}$.

$$B^{-1} = \begin{bmatrix} a & 0 & 0 \\ 0 & b^{-1} & 0 \\ 0 & 0 & a^{-1}b^{-1}c^{-1} \end{bmatrix}$$

[3]

(c) Let $\mathbf{C} = \begin{bmatrix} 4 & 3 & 2 & x \\ 0 & x & 3 & x \\ 0 & 0 & 1/x & x \\ 0 & 0 & 0 & x/2 \end{bmatrix}$. Determine $\det(\mathbf{C}^{-1})$.

$$\det(C^{-1}) = \frac{1}{\det(C)} = \frac{1}{(4)(\cancel{x})(\cancel{\tfrac{1}{x}})(\tfrac{x}{2})} = \boxed{\frac{1}{2x}}$$

[4]

Question 2: Compute the determinants of the following matrices. Show work or provide explanation to support your answers:

(a) $\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{matrix} 1 & 2 \\ 4 & 5 \\ 7 & 8 \end{matrix}$

$\det(A) = 45+84+96-105-48-72$

$= \boxed{0}$

[2]

(b) $\mathbf{B} = \begin{bmatrix} 2 & 1 & 6 & 2 \\ 3 & -2 & 4 & 1 \\ 2 & 1 & 6 & 2 \\ 3 & 5 & 2 & 4 \end{bmatrix}$

$\xrightarrow{R_1=(-1)r_3+r_1} B' = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 3 & -2 & 4 & 1 \\ 2 & 1 & 6 & 2 \\ 3 & 5 & 2 & 4 \end{bmatrix}$

$\therefore \det(B) = \det(B') = \boxed{0}$

[3]

(c) $\mathbf{C} = \begin{bmatrix} -2 & 1 & 4 & -1 \\ 1 & 0 & -1 & 2 \\ 5 & -1 & 2 & 1 \\ 0 & 0 & 3 & -1 \end{bmatrix}$

$\det(C) = \begin{vmatrix} -2 & 1 & 4 & -1 \\ 1 & 0 & -1 & 2 \\ 5 & -1 & 2 & 1 \\ 0 & 0 & 3 & -1 \end{vmatrix}$

$R_1 = 2r_2 + r_1$
$R_3 = -5r_2 + r_3$

$= \begin{vmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & -1 & 2 \\ 0 & -1 & 7 & -9 \\ 0 & 0 & 3 & -1 \end{vmatrix}$

$r_1 \leftarrow r_2$:

$= - \begin{vmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 2 & 3 \\ 0 & -1 & 7 & -9 \\ 0 & 0 & 3 & -1 \end{vmatrix}$

$r_3 \leftrightarrow r_4$:

$= \begin{vmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 3 & -1 \\ 0 & -1 & 7 & -9 \end{vmatrix}$

$R_4 = r_2 + r_4$

$= \begin{vmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 3 & -1 \\ 0 & 0 & 9 & -6 \end{vmatrix}$

$= (1)(1)[-18+9] = \boxed{-9}$

[5]

Question 3: Suppose $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 7$. Determine

$$\begin{vmatrix} 2d & 2e & 2f \\ a & b & c \\ g+3a & h+3b & i+3c \end{vmatrix}$$

$B = \begin{bmatrix} 2d & 2e & 2f \\ a & b & c \\ g+3a & h+3b & i+3c \end{bmatrix}$ is obtained from $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$

by (i) $r_1 \leftrightarrow r_2$, then (ii) $R_1 = 2r_1$, then (iii) $R_3 = 3r_2 + r_3$.

$\therefore \det(B) = (-1)(2)(1) \det(A)$

$= (-2)(7)$

$= \boxed{-14}$

[4]

Question 4: Let **A** be a 3×3 matrix each of whose entries is 1 or 0. What is the largest possible value for $\det(\mathbf{A})$?

Answer: 2!

Let $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$, so $\det(A) = aei + bfg + cdh - ceg - afh - bdi$

by the arrow method. Notice this shows $\det(A) \leq 1+1+1-0-0-0 = 3$

Now select a, e, i, b, f, g to be 1's; c, d, h to be 0. This gives $\det(A) = 2$.

$\det(A)$ can never be 3, since this would require $aei = bfg = cdh = 1$, implying all entries are 1's, i.e. all rows identical, but that would give $\det(A) = 0$.

[4]

Question 5: Use Cramer's rule to determine the value of x_4 in the following system of equations:

$$-x_1 + x_3 = 0$$
$$x_2 - 5x_3 + 3x_4 = 1$$
$$x_2 + 3x_4 = 0$$
$$2x_2 - 4x_3 + x_4 = 1$$

$A = \begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & 1 & -5 & 3 \\ 0 & 1 & 0 & 3 \\ 0 & 2 & -4 & 1 \end{bmatrix}$; $\det(A) = (-1)\left[(1)(0+12) + (5)(1-6) + (3)(-4-0)\right]$ } using col. 1

$= 25$

$A_4 = \begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & 1 & -5 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & -4 & 1 \end{bmatrix}$; $\det(A_4) = (-1)\left[(-1)(-5+4)\right] = -1$ } again using col. 1

$\therefore \quad x_4 = \dfrac{\det(A_4)}{\det(A)} = \boxed{\dfrac{-1}{25}}$

[7]

Question 6: Suppose **A** is 3×3 with $\det(\mathbf{A}) = 4$. Determine $\det[(2\mathbf{A})(-3\mathbf{A})^T]$.

$\det\left[(2A)(-3A)^T\right] = \det(2A)\det\left((-3A)^T\right)$

$= \det(2A)\det(-3A)$

$= 2^3\det(A)\,(-3)^3\det(A)$

$= (2)^3(4)(-3)^3(4)$

$= \boxed{-3456}$

[3]

Question 7: For this question use the vectors

$$\mathbf{u} = (0, -1, 0), \quad \mathbf{v} = (1, 1, 3), \quad \mathbf{w} = (2, 1, 2)$$

(a) Compute $\|2\mathbf{u} - 3(\mathbf{v} - \mathbf{w})\| = \| 2(0,-1,0) - 3((1,1,3) - (2,1,2)) \|$

$$= \| (0,-2,0) - 3(-1,0,1) \|$$
$$= \| (3,-2,-3) \|$$
$$= \sqrt{9+4+9}$$
$$= \boxed{\sqrt{22}}$$

[2]

(b) Determine the angle between $\mathbf{v}$ and $\mathbf{w}$ (if giving a calculator answer state units.)

$$\vec{v} \cdot \vec{w} = \|\vec{v}\| \, \|\vec{w}\| \cos \theta$$

$$\Rightarrow \quad \theta = \cos^{-1}\left(\frac{\vec{v} \cdot \vec{w}}{\|\vec{v}\| \, \|\vec{w}\|} \right)$$

$$= \cos^{-1}\left(\frac{(1,1,3)\cdot(2,1,2)}{\sqrt{1+1+9}\,\sqrt{4+1+4}} \right) = \cos^{-1}\left(\frac{9}{\sqrt{99}} \right) = \boxed{\cos^{-1}\left(\frac{3}{\sqrt{11}} \right)}$$

[2]

(c) Find a unit vector pointing in the same direction as $\mathbf{u} - \mathbf{v} + \mathbf{w}$.

$$\vec{u} - \vec{v} + \vec{w} = (0,-1,0) - (1,1,3) + (2,1,2) = (1,-1,-1)$$

$\therefore$ desired unit vector is $\dfrac{(1,-1,-1)}{\|(1,-1,-1)\|} = \dfrac{(1,-1,-1)}{\sqrt{3}} = \boxed{\left(\frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}} \right)}$

[3]

(d) Is it possible to express any vector $\mathbf{a} = (a_1, a_2, a_3)$ as a linear combination of $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$? That is, for each vector $\mathbf{a}$ can one always find scalars c_1, c_2 and c_3 so that $\mathbf{a} = c_1\mathbf{u} + c_2\mathbf{v} + c_3\mathbf{w}$?

Yes! $\vec{a} = c_1\vec{u} + c_2\vec{v} + c_3\vec{w}$ iff $\begin{bmatrix} 0 & 1 & 2 \\ -1 & 1 & 1 \\ 0 & 3 & 2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$,

and this has a unique solution iff $\det\left(\begin{bmatrix} 0 & 1 & 2 \\ -1 & 1 & 1 \\ 0 & 3 & 2 \end{bmatrix} \right) \neq 0$,

and $\det\left(\begin{bmatrix} 0 & 1 & 2 \\ -1 & 1 & 1 \\ 0 & 3 & 2 \end{bmatrix} \right) = 0(2-3) - 1(-2-0) + 2(-3-0) = -4 \neq 0$.

[3]