Question 1: Solve the following system of equations **using either Gaussian or Gauss-Jordan elimination** (no credit will be given for using any other method). Use proper notation to clearly state the row operations used at each step and clearly state the final solution.

$$2x + 3y - z = -2$$
$$x - y + z = 8$$
$$3x - 2y - 9z = 9$$

$$\begin{bmatrix} 2 & 3 & -1 & -2 \\ 1 & -1 & 1 & 8 \\ 3 & -2 & -9 & 9 \end{bmatrix}$$

$r_1 \leftrightarrow r_2$:
$$\begin{bmatrix} 1 & -1 & 1 & 8 \\ 2 & 3 & -1 & -2 \\ 3 & -2 & -9 & 9 \end{bmatrix}$$

$R_2 = (-2)r_1 + r_2$:
$R_3 = (-3)r_1 + r_3$:
$$\begin{bmatrix} 1 & -1 & 1 & 8 \\ 0 & 5 & -3 & -18 \\ 0 & 1 & -12 & -15 \end{bmatrix}$$

$r_2 \leftrightarrow r_3$:
$$\begin{bmatrix} 1 & -1 & 1 & 8 \\ 0 & 1 & -12 & -15 \\ 0 & 5 & -3 & -18 \end{bmatrix}$$

$R_3 = (-5)r_2 + r_3$:
$$\begin{bmatrix} 1 & -1 & 1 & 8 \\ 0 & 1 & -12 & -15 \\ 0 & 0 & 57 & 57 \end{bmatrix}$$

$R_3 = \left(\frac{1}{57}\right)r_3$:
$$\begin{bmatrix} 1 & -1 & 1 & 8 \\ 0 & 1 & -12 & -15 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$R_2 = (12)r_3 + r_2$;
$R_1 = (-1)r_3 + r_1$:
$$\begin{bmatrix} 1 & -1 & 0 & 7 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$R_1 = (1)r_2 + r_1$:
$$\begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$\therefore \ (x, y, z) = (4, -3, 1)$.

[10]

Question 2: Gauss-Jordan elimination was used to solve a sytem of 4 equations in the 5 variable $x_1, x_2, \ldots, x_5$, resulting in the following RREF matrix:

$$\begin{bmatrix} 1 & 2 & 0 & 0 & 3 & 2 \\ 0 & 0 & 1 & 0 & -1 & 4 \\ 0 & 0 & 0 & 1 & -2 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

State the general solution to the system of equations making proper use of any required parameters.

x_1, x_3, x_4 are leading variables; x_2 & x_5 are free.

Let $x_2 = r$, $x_5 = t$:

row 3 $\Rightarrow$ $x_4 = 3 + 2x_5 = 3 + 2t$

row 2 $\Rightarrow$ $x_3 = 4 + x_5 = 4 + t$

row 1 $\Rightarrow$ $x_1 = 2 - 2x_2 - 3x_5 = 2 - 2r - 3t$

$\therefore$ solution is $(2-2r-3t, r, 4+t, 3+2t, t)$ where r, t are any real numbers.

[4]

Question 3: Determine a relationship between a, b and c so that the following system has no solutions:

$$x + y = 2$$
$$y + z = 2$$
$$x + z = 2$$
$$ax + by + cz = 0$$

$$\begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 1 & 1 & 2 \\ 1 & 0 & 1 & 2 \\ a & b & c & 0 \end{bmatrix}$$

$R_3 = (-1)r_1 + r_3$;

$R_4 = (-a)r_1 + r_4$:

$$\begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 1 & 1 & 2 \\ 0 & -1 & 1 & 0 \\ 0 & (-a+b) & c & -2a \end{bmatrix}$$

$R_3 = (+1)r_2 + r_3$:

$R_4 = (a-b)r_2 + r_4$:

$$\begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & (a-b+c) & -2b \end{bmatrix}$$

$R_3 = (\frac{1}{2})r_3$:

$$\begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & (a-b+c) & -2b \end{bmatrix}$$

$R_4 = (-a+b-c)r_3 + r_4$:

$$\underbrace{\begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & (-a-b-c) \end{bmatrix}}$$

System has no solutions if $-a-b-c \neq 0$,

i.e $\boxed{a+b+c \neq 0}$

[6]

Question 4: For this problem use the following matrices to carry out the indicated computations, if possible. If a given statement is not defined then state "not defined":

$$\mathbf{A} = \begin{bmatrix} -3 & 1 \\ 2 & -1 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 0 & 4 \\ -2 & 5 \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} 5 & 0 \\ -1 & 4 \\ 3 & 3 \end{bmatrix} \quad \mathbf{D} = \begin{bmatrix} 1 & 0 & -3 \\ -2 & 5 & -1 \end{bmatrix} \quad \mathbf{E} = \begin{bmatrix} 1 & 4 & -5 \\ -2 & 1 & -3 \\ 0 & 2 & 6 \end{bmatrix}$$

(a) $(\mathbf{A}-\mathbf{B})\mathbf{C}^{\mathsf{T}} = \begin{bmatrix} -3 & -3 \\ 4 & -6 \end{bmatrix}\begin{bmatrix} 5 & -1 & 3 \\ 0 & 4 & 3 \end{bmatrix} = \begin{bmatrix} -15 & -9 & -18 \\ 20 & -28 & -6 \end{bmatrix}$.

[2]

(b) $\operatorname{tr}(\mathbf{DC}-5\mathbf{I}_2) = tr\left(\begin{bmatrix} 1 & 0 & -3 \\ -2 & 5 & -1 \end{bmatrix}\begin{bmatrix} 5 & 0 \\ -1 & 4 \\ 3 & 3 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}\right)$

$= tr\left(\begin{bmatrix} -4 & -9 \\ -18 & 17 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}\right)$

$= tr\begin{pmatrix} -9 & -9 \\ -18 & 12 \end{pmatrix} = \boxed{3}$

[2]

(c) **CABED**

3x2 2x2 2x2 3x3 2x3

not defined

[2]

(d) $(\mathbf{BA})^{-1} - \mathbf{A}^{-1}\mathbf{B}^{-1}$

$= A^{-1}B^{-1} - A^{-1}B^{-1}$

$= 0$

[2]

(e) Find a matrix $\mathbf{X}$ so that $(\mathbf{AX})^{-1} = \mathbf{B}$.

$(AX)^{-1} = B \Rightarrow AX = B^{-1}$

$\Rightarrow X = A^{-1}B^{-1} = (BA)^{-1} = \left(\begin{bmatrix} 0 & 4 \\ -2 & 5 \end{bmatrix}\begin{bmatrix} -3 & 1 \\ 2 & -1 \end{bmatrix}\right)^{-1} = \begin{bmatrix} 8 & -4 \\ 16 & -7 \end{bmatrix}^{-1}$

$\therefore X = \dfrac{1}{(8)(-7)-(-4)(16)}\begin{bmatrix} -7 & 4 \\ -16 & 8 \end{bmatrix} = \begin{bmatrix} -7/8 & 1/2 \\ -2 & 1 \end{bmatrix}$

[2]

Question 5:

(a) Determine $\mathbf{A}^{-1}$ where $\mathbf{A} = \begin{bmatrix} 2 & 3 & 1 \\ 3 & 3 & 1 \\ 2 & 4 & 1 \end{bmatrix}$

$$\left[\begin{array}{ccc|ccc} 2 & 3 & 1 & 1 & 0 & 0 \\ 3 & 3 & 1 & 0 & 1 & 0 \\ 2 & 4 & 1 & 0 & 0 & 1 \end{array}\right]$$

$r_1 \leftrightarrow r_2$:
$$\left[\begin{array}{ccc|ccc} 3 & 3 & 1 & 0 & 1 & 0 \\ 2 & 3 & 1 & 1 & 0 & 0 \\ 2 & 4 & 1 & 0 & 0 & 1 \end{array}\right]$$

$R_1 = (-1) r_2 + r_1$:
$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & 0 \\ 2 & 3 & 1 & 1 & 0 & 0 \\ 2 & 4 & 1 & 0 & 0 & 1 \end{array}\right]$$

$R_2 = (-2) r_1 + r_2$:
$R_3 = (-2) r_1 + r_3$:
$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 3 & 1 & 3 & -2 & 0 \\ 0 & 4 & 1 & 2 & -2 & 1 \end{array}\right]$$

$r_2 \leftrightarrow r_3$:
$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 4 & 1 & 2 & -2 & 1 \\ 0 & 3 & 1 & 3 & -2 & 0 \end{array}\right]$$

$R_2 = (-1) r_3 + r_2$:
$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 3 & 1 & 3 & -2 & 0 \end{array}\right]$$

$R_3 = (-3) r_2 + r_3$:
$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 6 & -2 & -3 \end{array}\right].$$

$$\therefore A^{-1} = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 6 & -2 & -3 \end{bmatrix}.$$

[7]

(b) Express the following system of equations as a matrix equation $\mathbf{Ax} = \mathbf{b}$ and use your result in part (a) to solve the system:

$$2x + 3y + z = 4$$
$$3x + 3y + z = 8$$
$$2x + 4y + z = 5$$

$$\begin{bmatrix} 2 & 3 & 1 \\ 3 & 3 & 1 \\ 2 & 4 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \\ 5 \end{bmatrix}$$

$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 6 & -2 & -3 \end{bmatrix} \begin{bmatrix} 4 \\ 8 \\ 5 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ -7 \end{bmatrix}$$

$$\therefore (x, y, z) = (4, 1, -7)$$

[3]

Question 6: Determine the values (if any) of the constants b and c so that the following matrix is invertible:

$$A = \begin{bmatrix} 0 & 1 & b \\ -1 & 0 & c \\ -b & -c & 0 \end{bmatrix}$$

A is invertible if and only if A reduces to I_3 using EROS:

$$\begin{bmatrix} 0 & 1 & b \\ -1 & 0 & c \\ -b & -c & 0 \end{bmatrix}$$

$r_1 \leftrightarrow r_2$: $\begin{bmatrix} -1 & 0 & c \\ 0 & 1 & b \\ -b & -c & 0 \end{bmatrix}$

$R_1 = (-1)r_1$: $\begin{bmatrix} 1 & 0 & -c \\ 0 & 1 & b \\ -b & -c & 0 \end{bmatrix}$

$R_3 = br_1 + r_2$: $\begin{bmatrix} 1 & 0 & -c \\ 0 & 1 & b \\ 0 & -c & -bc \end{bmatrix}$

$R_3 = cr_2 + r_3$: $\begin{bmatrix} 1 & 0 & -c \\ 0 & 1 & b \\ 0 & 0 & 0 \end{bmatrix}$ ← A does not reduce to I_3 for any choice of b & C.

[6]

Question 7: Explain why a square matrix that has a row of zeros can never have an inverse.

If A is $n \times n$ and has a row of zeros, then the row of zeros will persist throughout the Gauss-Jordan procedure, so that the RREF of A will also have a row of zeros. That is, the RREF of A is not I_n, so A is not invertible.

[4]
