Math 141 - Matrix Algebra for Engineers

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Matrices and Matrix Operations

Matrices

- Matrices arise in other areas of mathematics and need not simply represent systems of linear equations.
- Matrices can encode the movement of points on a computer display, the interaction between resources and products in an economy, the probabilities associated with the state of a system at different points in time, etc. Let's examine the algebraic properties of matrices viewed purely as mathematical object.
- **Important:** in the following discussion, the matrices are not necessarily associated with systems of linear equations. They should be interpreted simply as rectangular arrays of real numbers.

Matrices

- **Definition:** A rectangular array of numbers consisting of m horizontal ***rows*** and n vertical ***columns***

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

  is called an $m \times n$ ***matrix*** (or matrix of ***size*** or ***dimension*** $m \times n$). For entry a_{ij}, i is the row subscript, while j is the column subscript.

- A general matrix is sometimes denoted $\left[a_{ij}\right]_{m\times n}$
- When referencing the size of a matrix or entries within the matrix, it's always **row first, column second**

Matrix Examples

- Let $B = \begin{bmatrix} 1 & 6/7 & 4 \\ 1/2 & -7 & 3 \end{bmatrix}$.

- The size of B is 2×3.

- A couple of entries of B are $b_{23} = 3, \;\; b_{21} = 1/2$.

- Notice: upper case letter to represent matrix, corresponding lowercase letter for entries.

Row and Column Vectors

- Let $\mathbf{p} = \begin{bmatrix} 3 & -4 & \pi \end{bmatrix}$.
- $\mathbf{p}$ is called a ***row matrix*** or ***row vector***. Here $p_1 = 3$, $p_2 = -4$ and $p_3 = \pi$.
- Let $\mathbf{q} = \begin{bmatrix} 0 \\ 2 \\ e \end{bmatrix}$.
- $\mathbf{q}$ is called a ***column matrix*** or ***column vector***. Here $q_1 = 0$, $q_2 = 2$ and $q_3 = e$.
- Notice: bold-face lower case letters used to represent vectors. For writing by hand, use $\vec{p}$, $\vec{q}$.

Example

Example: Construct $[a_{ij}]_{4\times 3}$ if $a_{ij} = \dfrac{1}{i+j}$.

Solution: Let $A = [a_{ij}]_{4\times 3}$. The entries of A are functions of their row and column positions; that is, $a_{ij} = 1/(i+j)$ is a function of the two variables $i = 1, 2, 3, 4$ and $j = 1, 2, 3$:

$$a_{11} = \frac{1}{1+1} = \frac{1}{2}\ ,\ a_{12} = \frac{1}{1+2} = \frac{1}{3}\ ,\ a_{13} = \frac{1}{1+3} = \frac{1}{4}\ ,$$

and so on. Computing all of the entries in this way we have

$$A = \begin{bmatrix} \frac{1}{1+1} & \frac{1}{1+2} & \frac{1}{1+3} \\ \frac{1}{2+1} & \frac{1}{2+2} & \frac{1}{2+3} \\ \frac{1}{3+1} & \frac{1}{3+2} & \frac{1}{3+3} \\ \frac{1}{4+1} & \frac{1}{4+2} & \frac{1}{4+3} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \\ \frac{1}{5} & \frac{1}{6} & \frac{1}{7} \end{bmatrix}\ .$$

Square Matrices

- A matrix with the same number n of rows and columns is called a ***square matrix*** of order n.
- Example,

$$A = \begin{bmatrix} 1 & -2 & 1 \\ -2 & 3 & 1 \\ 5 & -7 & -3 \end{bmatrix}$$

- Here the entries $a_{11} = 1$, $a_{22} = 3$, and $a_{33} = -3$ (reading from upper-left to lower-right) form the ***main diagonal*** of A.

Equality of Matrices

- The matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are said to be ***equal*** if A and B have the same size and $a_{ij} = b_{ij}$ for each i and j.

- **Example:** $\begin{bmatrix} 2 & 3 \\ 8 & 7 \end{bmatrix} = \begin{bmatrix} 1+1 & 3 \\ 16/2 & 9-2 \end{bmatrix}$

- **Example:** $\begin{bmatrix} 2 \\ 2 \end{bmatrix} \neq \begin{bmatrix} 2 & 2 \end{bmatrix}$ since the sizes of are not the same: $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$ is size 2×1 while $\begin{bmatrix} 2 & 2 \end{bmatrix}$ is size 1×2.

Matrix Addition

- Let $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{m \times n}$. Then

$$A + B = [a_{ij} + b_{ij}]_{m \times n} \ .$$

- That is, provided A and B are the same size, the matrix $C = A + B$ is simply the $m \times n$ matrix formed by adding the corresponding entries of A and B: $c_{ij} = a_{ij} + b_{ij}$.
- **Example:** Let

$$A = \begin{bmatrix} 3 & 0 & -2 \\ 4 & 3 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} -11 & 1 & 12 \\ 7 & -2 & 13 \end{bmatrix}.$$

Then

$$A+B = \begin{bmatrix} 3+(-11) & 0+1 & (-2)+12 \\ 4+7 & 3+(-2) & 1+13 \end{bmatrix} = \begin{bmatrix} -8 & 1 & 10 \\ 11 & 1 & 14 \end{bmatrix}$$

Matrix Subtraction

- If A and B are the same size, the matrix $C = A - B$ is the $m \times n$ matrix formed by subtracting the entries of B from the corresponding entries of A: $c_{ij} = a_{ij} - b_{ij}$.
- **Example:** Let

$$A = \begin{bmatrix} 3 & 0 & -2 \\ 4 & 3 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} -11 & 1 & 12 \\ 7 & -2 & 13 \end{bmatrix}.$$

Then

$$A - B = \begin{bmatrix} 3-(-11) & 0-1 & (-2)-12 \\ 4-7 & 3-(-2) & 1-13 \end{bmatrix} = \begin{bmatrix} 14 & -1 & -14 \\ -3 & 5 & -12 \end{bmatrix}$$

Scalar Multiplication

- Let $A = [a_{ij}]_{m \times n}$ be a matrix and let k be a real number (a ***scalar***). Then $C = kA$ is the matrix with entry $c_{ij} = ka_{ij}$.
- **Example:** For

$$A = \begin{bmatrix} 1 & 6 \\ 4 & 2 \end{bmatrix},$$

multiplication by the scalar $1/2$ gives

$$\frac{1}{2}A = \frac{1}{2}\begin{bmatrix} 1 & 6 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} (1/2)(1) & (1/2)(6) \\ (1/2)(4) & (1/2)(2) \end{bmatrix} = \begin{bmatrix} 1/2 & 3 \\ 2 & 1 \end{bmatrix}.$$

Matrix Multiplication

- **Definition:** Let $A = [a_{ij}]_{m \times p}$ and $B = [b_{ij}]_{p \times n}$. Then we define the product $C = AB$ as the matrix with entries

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} + \cdots + a_{ip}b_{pj} \ .$$

- Notice:
  - *(i)* $a_{i1}, a_{i2}, a_{i3}, \ldots, a_{ip}$ are the elements of row i of A, while $b_{1j}, b_{2j}, b_{3j}, \ldots, b_{pj}$ are the elements of column j of B.
  - *(ii)* For matrix multiplication to be defined, the number of columns of A must be the same as the number of rows of B.

Matrix Multiplication Example

Let

$$A = \begin{bmatrix} 2 & 3 \\ 1 & -2 \\ 6 & 4 \end{bmatrix}, \qquad B = \begin{bmatrix} 3 & 7 \\ -2 & 5 \end{bmatrix}.$$

Then

$$AB = \begin{bmatrix} (2)(3) + (3)(-2) & (2)(7) + (3)(5) \\ (1)(3) + (-2)(-2) & (1)(7) + (-2)(5) \\ (6)(3) + (4)(-2) & (6)(7) + (4)(5) \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 29 \\ -1 & -3 \\ 10 & 62 \end{bmatrix}.$$

Notice A is size 3×2, B is size 2×2, and the product AB is size 2×2.

Matrix Multiplication: Another way to think about it

- Multiplying matrices can be described in terms of a certain type of product: Let $\mathbf{r}$ and $\mathbf{c}$ be the row and column matrices

$$\mathbf{r} = \begin{bmatrix} r_1 & r_2 & \cdots & r_p \end{bmatrix} \text{ and } \mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_p \end{bmatrix} .$$

- The expression $r_1 c_1 + r_2 c_2 + \cdots + r_p c_p$ is called the ***dot product*** of $\mathbf{r}$ and $\mathbf{c}$.
- The matrix product $D = AB$ is the matrix which has entries

$$d_{ij} = \text{ dot product of row } i \text{ of } A \text{ and column } j \text{ of } B \text{ .}$$

Matrix Multiplication: Yet another way to think about it

- For the matrix product AB, partition B into column vectors:

$$B = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_n \end{bmatrix}$$

- Then

$$AB = A \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_n \end{bmatrix} = \begin{bmatrix} A\mathbf{b}_1 & A\mathbf{b}_2 & \cdots & A\mathbf{b}_n \end{bmatrix}$$

Matrix Multiplication: Yet another way to think about it

- Similarly, partition A into row vectors:

$$A = \begin{bmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \vdots \\ \mathbf{a}_m \end{bmatrix}$$

- Then

$$AB = \begin{bmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \vdots \\ \mathbf{a}_m \end{bmatrix} B = \begin{bmatrix} \mathbf{a}_1 B \\ \mathbf{a}_2 B \\ \vdots \\ \mathbf{a}_m B \end{bmatrix}$$

Transpose of a Matrix

- Suppose A is an $m \times n$ matrix. The ***transpose*** of A, denoted A^{T}, is the matrix of size $n \times m$ obtained by interchanging the rows and columns of A.
- **Example:** For

$$A = \begin{bmatrix} 1 & -2 & 1 \\ -2 & 3 & 1 \\ 5 & -7 & -3 \end{bmatrix},$$

$$A^{\mathrm{T}} = \begin{bmatrix} 1 & -2 & 5 \\ -2 & 3 & -7 \\ 1 & 1 & -3 \end{bmatrix}.$$

Matrix Transpose Example

- Let

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \quad \Big\} \text{ size } 2 \times 3$$

then

$$B^{\mathsf{T}} = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} \quad \Bigg\} \text{ size } 3 \times 2$$

- Using this last example, notice that

$$(B^{\mathsf{T}})^{\mathsf{T}} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = B \,.$$

- This turns out to be true in general: for any matrix A, $(A^{\mathsf{T}})^{\mathsf{T}} = A$.

Trace of a Matrix

- If A is a square matrix, the ***trace*** of A, written $\mathrm{tr}(A)$ is the sum of the entries on the main diagonal of A.

- **Example:** For

$$A = \begin{bmatrix} 1 & -2 & 1 \\ -2 & 3 & 1 \\ 5 & -7 & -3 \end{bmatrix},$$

$$\mathrm{tr}(A) = 1 + 3 + (-3) = 1$$

Systems of Linear Equations as Matrix Products

- Matrix multiplication can be used to express systems of linear equations.
- For example, the system

$$\begin{aligned} 5x - 2y + z &= -2 \\ -x + 11y - 13z &= 6 \\ x + y + z &= 1 \end{aligned}$$

  is equivalent to

$$\begin{bmatrix} 5 & -2 & 1 \\ -1 & 11 & -13 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 \\ 6 \\ 1 \end{bmatrix}$$

Systems of Linear Equations as Matrix Products

That is …

$$\begin{bmatrix} 5 & -2 & 1 \\ -1 & 11 & -13 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 \\ 6 \\ 1 \end{bmatrix}$$

is equivalent to

$$A\mathbf{x} = \mathbf{b}$$

where A is the coefficient matrix of the system, and

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} -2 \\ 6 \\ 1 \end{bmatrix}$$