Math 141 - Matrix Algebra for Engineers

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Linear Independence, Coordinates and Basis

Summary of Important Relationships

Theorem

Suppose $\mathbf{A}$ is an $n \times n$ matrix. The following statements are equivalent (that is, they are either all true, or all false):

1. $\mathbf{A}$ is invertible (i.e. $\mathbf{A}^{-1}$ exists.)
2. $\mathbf{Ax} = \mathbf{0}$ has only the trivial solution.
3. The RREF form of $\mathbf{A}$ is $\mathbf{I_n}$.
4. $\mathbf{A}$ can be expressed as a product of elementary matrices.
5. $\mathbf{Ax} = \mathbf{b}$ has a unique solution for every $n \times 1$ matrix $\mathbf{b}$.
6. $\det(\mathbf{A}) \neq 0$.

Recap: Linear Independence

- **Definition**

  Let

  $$S = \{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_r}\}$$

  be a non-empty subset of a vector space V.
  S is linearly independent if the only solutions to

  $$k_1\mathbf{v_1} + k_2\mathbf{v_2} + \cdots + k_r\mathbf{v_r} = \mathbf{0}$$

  is

  $$k_1 = k_2 = \cdots = k_r = 0$$

- This says: $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_r}$ are linearly independent if it is not possible to express any one vector as a linear combination of the others .
- If S is not linearly independent, say that S (or the vectors of S) are linearly dependent.

Linear Independence of Functions

- Consider $F(-\infty, \infty) = \{f(x) \mid f(x) \text{ has domain } \mathbb{R}\}$, and let
$$S = \{f_1(x), f_2(x), \ldots, f_n(x)\}$$

- S is linearly independent if whenever
$$k_1 f_1(x) + k_2 f_2(x) + \cdots + k_n f_n(x) = 0 \leftarrow \text{the zero function}$$
then $k_1 = k_2 = \cdots = k_n = 0$.

- Problem: testing for linear independence of functions using the definition is tricky. There is another way

Wronskian Determinants

Definition:
Suppose $f_1(x), f_2(x), \ldots, f_n(x)$ are $(n-1)$-times differentiable on $\mathbb{R}$. The determinant

$$W(x) = \begin{vmatrix} f_1(x) & f_2(x) & \cdots & f_n(x) \\ f_1'(x) & f_2'(x) & \cdots & f_n'(x) \\ f_1''(x) & f_2''(x) & \cdots & f_n''(x) \\ \vdots & \vdots & \vdots & \vdots \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \cdots & f_n^{(n-1)}(x) \end{vmatrix}$$

is called the Wronskian of $f_1(x), f_2(x), \ldots, f_n(x)$.

Wronskians and Linear Independence

Theorem

Suppose $f_1(x), f_2(x), \ldots, f_n(x)$ have $(n-1)$ continuous derivatives on $\mathbb{R}$. If $W(x) \neq 0$ for at least one $x \in \mathbb{R}$ then $\{f_1(x), f_2(x), \ldots, f_n(x)\}$ is a linearly independent set.

Caution: if $W(x) = 0$ for every x then no conclusion can be drawn about linear independence.

Wronskians: Example

Example

Is $S = \{t, e^{-t}, e^t\}$ a linearly independent set?

Solution

Here $f_1(t) = t$, $f_2(t) = e^{-t}$, $f_3(t) = e^t$.

$$W(t) = \begin{vmatrix} t & e^{-t} & e^t \\ 1 & -e^{-t} & e^t \\ 0 & e^{-t} & e^t \end{vmatrix} = t(-e^{-t}e^t - e^{-t}e^t) - 1(e^{-t}e^t - e^{-t}e^t) = -2t$$

Since $W(t) = -2t \neq 0$ for at least one real t is follows that $S = \{t, e^{-t}, e^t\}$ is a linearly independent set.

Proof of Theorem

To show that $W(x) \neq 0$ implies linear independence of $f_1(x), f_2(x), \ldots, f_n(x)$, we will prove the contrapositive:

If $f_1(x), f_2(x), \ldots, f_n(x)$ are linearly dependent, then $W(x)$ must be zero.

Suppose $f_1(x), f_2(x), \ldots, f_n(x)$ are linearly dependent. Then there are scalars (not all zero) so that

$$k_1 f_1(x) + k_2 f_2(x) + \cdots + k_n f_n(x) = 0 \leftarrow \text{the zero function}$$

Differentiating both sides of this equation $(n-1)$ times gives a system of equations:

Proof Continued

$$\begin{bmatrix} f_1(x) & f_2(x) & \cdots & f_n(x) \\ f_1'(x) & f_2'(x) & \cdots & f_n'(x) \\ f_1''(x) & f_2''(x) & \cdots & f_n''(x) \\ \vdots & \vdots & \vdots & \vdots \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \cdots & f_n^{(n-1)}(x) \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ \vdots \\ k_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Now $W(x)$ (= the determinant of the square matrix) must be zero, since otherwise this system would have the unique solution $k_1 = k_2 = \cdots = k_n = 0$, contrary to our assumption.

Coordinates and Basis in Vector Spaces

- We saw that R^3 consists of all **linear combinations**

$$\mathbf{u} = a\,\mathbf{i} + b\,\mathbf{j} + c\,\mathbf{k}$$

  where a, b and c are scalars.

- Here $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ is a basis for R^3 .

- The scalars a, b, c are the coordinates of $\mathbf{u}$.

- We wish to generalize this idea.

Basis for a vector space

Definition:

Suppose

$$S = \{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}\}$$

is a finite subset of a vector space V.

S is called a basis for V if

1. S is linearly independent, and
2. $V = \text{span}(S)$

Basis Examples

- $\mathbf{i} = (1, 0, 0),\ \mathbf{j} = (0, 1, 0),\ \mathbf{k} = (0, 0, 1)$ is called the standard basis for R^3 .

- More generally, the set of n-tuples

  $$\mathbf{e_1} = (1, 0, 0, \ldots, 0), \mathbf{e_2} = (0, 1, 0, \ldots, 0), \ldots, \mathbf{e_n} = (0, 0, 0, \ldots, 1)$$

  is called the standard basis for R^n .

- $S = \{1, x, x^2, \ldots, x^n\}$ is a basis for P_n .

Coordinates Relative to a Basis.

Theorem

If $S = \{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}\}$ is a basis for a vector space V, then every vector $\mathbf{v}$ in V has a unique representation as a linear combination of vectors from S:

$$\mathbf{v} = c_1\mathbf{v_1} + c_2\mathbf{v_2} + \cdots + c_n\mathbf{v_n}$$

Definition

If $S = \{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}\}$ is a basis for a vector space V and

$$\mathbf{v} = c_1\mathbf{v_1} + c_2\mathbf{v_2} + \cdots + c_n\mathbf{v_n} \ ,$$

the scalars $c_1, \ldots, c_n$ are called the coordinates of $\mathbf{v}$ relative to S, and we write

$$(\mathbf{v})_S = (c_1, c_2, \ldots, c_n)$$

Dimension of a Vector Space

Definition

Let V be a vector space with basis $S = \{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}\}$. The dimension of V is n, and we write

$$\dim(V) = n$$

(the zero vector space is defined to have dimension zero.)

For this definition to be unambiguous:

Theorem

Let V be a vector space with basis $S = \{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}\}$. Then every basis for V contains n vectors.