Math 141 - Matrix Algebra for Engineers

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Gaussian and Gauss-Jordan Elimination

Elementary Row Operations

Elementary Row Operations (EROs) change the form of a matrix without changing the solution of the corresponding system of equations:

1. Interchange any two rows.

   Notation to interchange, say, rows 2 and 3: $r_2 \leftrightarrow r_3$.

2. Multiply a row through by a non-zero constant.

   Notation to multiply, say, row 2 by -3 resulting in a new row 2: $R_2 = (-3)r_2$.

3. Add a constant of one row to another row.

   Notation to add, say, -7 times row 2 to row 3 to get a new row 3: $R_3 = (-7)r_2 + r_3$.

Row Echelon Form

A matrix is said to be in **Row Echelon Form (REF)** if

1. The first non-zero entry in any row is a 1 (called a **leading 1**).
2. The leading 1 in any row is located to the right of the leading 1 of any row above.
3. Any rows consisting entirely of zeros are at the bottom of the matrix.

Row Echelon Form Example

- A matrix in REF:

$$\begin{bmatrix} 1 & -4 & 2 & -1 & 3 \\ 0 & 0 & 1 & 3 & -2 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

- A matrix not in REF:

$$\begin{bmatrix} 1 & -4 & 2 & -1 & 3 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 1 & 3 & -2 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Reduced Row Echelon Form

A matrix is said to be in **Reduced Row Echelon Form (RREF)** if, in addition to being in REF,

4. Any column containing a leading 1 has zeros elsewhere in the column.

Reduced Row Echelon Form Examples

- In RREF? $\begin{bmatrix} 1 & 1 & 0 & 5 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ Yes!
- In RREF? $\begin{bmatrix} 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 3 \\ 1 & 0 & 0 & 0 \end{bmatrix}$ No!
- In RREF? $\begin{bmatrix} 1 & 1 & 0 & 5 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ No!

Gaussian Elimination Algorithm

A step-by-step procedure which transforms an augmented matrix into REF. To put a matrix into REF:

1. Interchange rows (if necessary) so that the first non-zero entry in the top row is located as far to the left as possible.
2. Use EROs to reduce the first entry in the top row to a leading 1. This can always be done by multiplying the top row by the reciprocal of the first entry in that row. A constant multiple of some other row can also be added to the top row to achieve this. Avoid introducing fractions if possible.
3. Now add suitable multiples of the top row to the rows below so that all entries below the leading 1 become zero.
4. Now, **without changing or using the top row**, go to step 1 and apply the procedure to the submatrix consisting of all rows below that containing the most recently used leading 1.
5. Proceed until there are no more rows.

Gauss-Jordan Elimination Algorithm

To put a matrix into RREF, first put it into REF using Gaussian elimination (called the **forward phase**), then perform an extra step (the **backward phase**):

6. Beginning with the last non-zero row and working upward, add suitable multiples of each row to the rows above to introduce zeros above the leading 1's.

Uniqueness of REF and RREF's

For a given starting augmented matrix,

- The REF is not unique: different choices of ERO's may result in different REF's. The solution to the system of equations will be the same however.
- The RREF is unique: there is one and only one RREF corresponding to a given starting augmented matrix, regardless of the ERO's used or order in which they are applied.

Example: System with Infinitely Many Solutions

Solve the system

$$\begin{aligned} 4x_2 + 3x_3 - 6x_4 &= -2 \\ -x_1 + x_2 + x_3 &= 2 \\ x_1 - 2x_2 - x_3 - 3x_4 &= -3 \end{aligned}$$

using Gauss-Jordan elimination.

Example: System with Infinitely Many Solutions

Set up augmented matrix:

$$\begin{bmatrix} 0 & 4 & 3 & -6 & -2 \\ -1 & 1 & 1 & 0 & 2 \\ 1 & -2 & -1 & -3 & -3 \end{bmatrix}$$

Now, reduce…

Example: System with Infinitely Many Solutions

$$r_1 \leftrightarrow r_3 : \quad \begin{bmatrix} 1 & -2 & -1 & -3 & -3 \\ -1 & 1 & 1 & 0 & 2 \\ 0 & 4 & 3 & -6 & -2 \end{bmatrix}$$

$$R_2 = r_1 + r_2 : \quad \begin{bmatrix} 1 & -2 & -1 & -3 & -3 \\ 0 & -1 & 0 & -3 & -1 \\ 0 & 4 & 3 & -6 & -2 \end{bmatrix}$$

$$R_2 = (-1)r_2 : \quad \begin{bmatrix} 1 & -2 & -1 & -3 & -3 \\ 0 & 1 & 0 & 3 & 1 \\ 0 & 4 & 3 & -6 & -2 \end{bmatrix}$$

Example: System with Infinitely Many Solutions

$$R_3 = (-4)r_2 + r_3 : \quad \begin{bmatrix} 1 & -2 & -1 & -3 & -3 \\ 0 & 1 & 0 & 3 & 1 \\ 0 & 0 & 3 & -18 & -6 \end{bmatrix}$$

$$R_3 = (1/3)r_3 : \quad \begin{bmatrix} 1 & -2 & -1 & -3 & -3 \\ 0 & 1 & 0 & 3 & 1 \\ 0 & 0 & 1 & -6 & -2 \end{bmatrix}$$

This completes the **forward phase** of Gauss-Jordan elimination.

The matrix is now in REF. Now for the backward phase…

Example: System with Infinitely Many Solutions

$$R_1 = r_3 + r_1 : \quad \begin{bmatrix} 1 & -2 & 0 & -9 & -5 \\ 0 & 1 & 0 & 3 & 1 \\ 0 & 0 & 1 & -6 & -2 \end{bmatrix}$$

$$R_1 = (2)r_2 + r_1 : \quad \begin{bmatrix} 1 & 0 & 0 & -3 & -3 \\ 0 & 1 & 0 & 3 & 1 \\ 0 & 0 & 1 & -6 & -2 \end{bmatrix}$$

This completes the **backward phase** of Gauss-Jordan elimination.

The matrix is now in RREF.

The variables associated with the leading 1's (x_1, x_2 and x_3) are called **leading variables.**

The remaining variables are called **free variables.**

Example: System with Infinitely Many Solutions

$$\left[\begin{array}{ccccc} 1 & 0 & 0 & -3 & -3 \\ 0 & 1 & 0 & 3 & 1 \\ 0 & 0 & 1 & -6 & -2 \end{array}\right]$$

Assigning parameters to the free variables and solving for the leading variables gives the **general solution**:

$$\begin{aligned} x_4 &= t \\ \text{so } x_3 &= -2 + 6t \ , \\ x_2 &= 1 - 3t \ , \\ x_1 &= -3 + 3t \end{aligned}$$

where t is any real number.

Alternatively: $(x_1, x_2, x_3, x_4) = (-3 + 3t, 1 - 3t, -2 + 6t, t)$.

Another Example: Infinitely Many Solutions

Suppose the RREF of a system is $\begin{bmatrix} 1 & -6 & 0 & 0 & 3 & -2 \\ 0 & 0 & 1 & 0 & 4 & 7 \\ 0 & 0 & 0 & 1 & 5 & 8 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

Here x_1, x_3, x_4 are leading variable, while x_2 and x_5 are free. Let

$$x_2 = r, x_5 = t \; .$$

Solving for x_1, x_3, x_4:

$$x_4 = 8 - 5t \; , x_3 = 7 - 4t \; , x_1 = -2 + 6r - 3t$$

So the general solution is

$$(x_1, x_2, x_3, x_4, x_5) = (-2 + 6r - 3t, r, 7 - 4t, 8 - 5t, t)$$

where r and t are any real numbers.

In Summary

Gaussian (or Gauss-Jordan) elimination will result in exactly one of the following three outcomes:

- Some row of the REF or RREF will have zeros in all but the right-most position. In this case the system is inconsistent (has no solution.) Otherwise...
- The REF (or RREF) will have fewer non-zero rows than variables. In this case the system is consistent with infinitely many solutions. Assign a different parameter to each of the free variables and solve for the leading variables in terms of the free variable parameters. Otherwise...
- The REF (or RREF) will have the same number of non-zero rows as variables and there is a single solution to the system. The system is consistent.

Homogeneous Linear Systems

Definition: A system of linear equations is called **homogeneous** if it has the form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= 0 \end{aligned}$$

Notice: A homogeneous system has least one solution:

$$x_1 = x_2 = \cdots = x_n = 0$$

This is called the **trivial solution.**

Homogeneous Linear Systems

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= 0 \end{aligned}$$

- Suppose the RREF of the system has r leading 1's and consequently r non-zero rows. Then it has $n - r$ free variables.
- Since $r \leq m$, $n - r \geq n - m$. So, if $n - m > 0$, that is, **if the number of variables exceeds the number of equations in the RREF**, then the number of free variables is greater than zero, and so **the system has infinitely many solutions.**