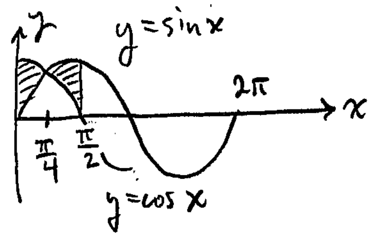
Question 1 [10 points]:

(a) Determine the area of the region in the first quadrant that is bounded between $y = \sin(x)$ and $y = \cos(x)$ over the interval $0 \le x \le \pi/2$.



$$A = \int_0^{\frac{\pi}{4}} \cos(x) - \sin(x)\, dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin x - \cos x \, dx$$

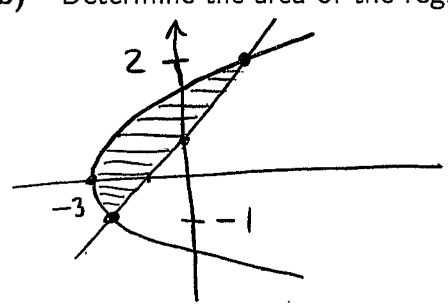
$$= \Big[\sin(x) + \cos(x)\Big]_0^{\frac{\pi}{4}} + \Big[-\cos(x) - \sin x\Big]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$

$$= \left(\tfrac{\sqrt{2}}{2} + \tfrac{\sqrt{2}}{2}\right) - (0+1) + (0-1) - \left(-\tfrac{\sqrt{2}}{2} - \tfrac{\sqrt{2}}{2}\right)$$

$$= \boxed{2(\sqrt{2}-1)}$$

[5]

(b) Determine the area of the region bounded between the curves $x = y^2 - 3$ and $y = x + 1$.



Intersection points: $y^2 - 3 = y - 1$

$y^2 - y - 2 = 0$

$(y-2)(y+1) =$

$y = 2 \quad ; \quad y = -1$

$\therefore\ x = 1 \quad ; \quad x = -2$

$$\therefore \quad A = \int_{y=-1}^{2} (y-1) - (y^2 - 3)\, dy$$

$$= \left[-\frac{y^3}{3} + \frac{y^2}{2} + 2y \right]_{-1}^{2}$$

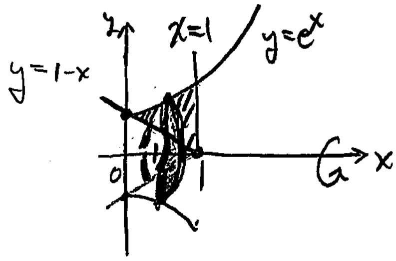
$$= \left(-\frac{8}{3} + 2 + 4\right) - \left(\frac{1}{3} + \frac{1}{2} - 2\right)$$

$$= \frac{-16 + 12 + 24 - 2 - 3 + 12}{6} = \frac{27}{6} = \boxed{\frac{9}{2}}$$

[5]

Question 2 [10 points]:

(a) The region in the first quadrant bounded by $y = e^x$, $y = 1 - x$, $x = 1$ and the y-axis is rotated about the x-axis. Determine the volume of the resulting solid. (The disk method would be best here.)



$$V = \pi \int_0^1 (e^x)^2 - (1-x)^2 \, dx$$

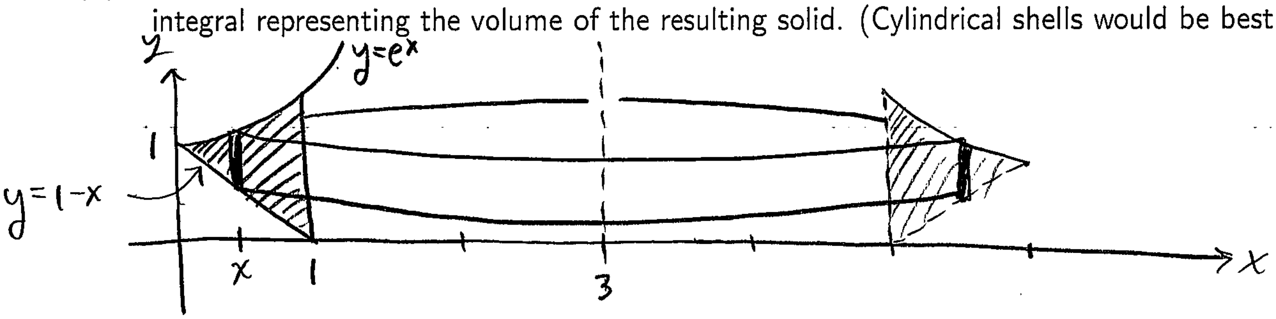
$$= \pi \left[\frac{e^{2x}}{2} + \frac{(1-x)^3}{3} \right]_0^1$$

$$= \pi \left[\left(\frac{e^2}{2} + 0 \right) - \left(\frac{1}{2} + \frac{1}{3} \right) \right]$$

$$= \boxed{\frac{\pi}{6} \left[3e^2 - 5 \right]}$$

[7]

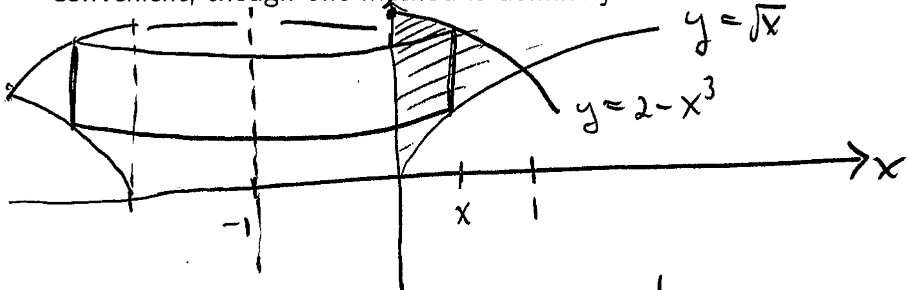
(b) The same region as in part (a) is rotated about the line $x = 3$. Set up but DO NOT EVALUATE the integral representing the volume of the resulting solid. (Cylindrical shells would be best here.)



$$\boxed{V = \int_0^1 2\pi (3-x) \left(e^x - (1-x) \right) dx.}$$

[3]

Question 3 [10 points]: The region in the first quadrant bounded by $y = 2 - x^3$, $y = \sqrt{x}$ and the y-axis is rotated about the line $x = -1$. Determine the volume of the resulting solid. (Use whichever method is most convenient, though one method is definitely better than the other in this case.)



By cylindrical shells: $V = \int_0^1 2\pi (x+1)(2-x^3 - x^{1/2})\,dx$

$$= 2\pi \int_0^1 2x - x^4 - x^{3/2} + 2 - x^3 - x^{1/2}\,dx$$

$$= 2\pi \left[x^2 - \frac{x^5}{5} - \frac{2}{5}x^{5/2} + 2x - \frac{x^4}{4} - \frac{2}{3}x^{3/2} \right]_0^1$$

$$= 2\pi \left[1 - \frac{1}{5} - \frac{2}{5} + 2 - \frac{1}{4} - \frac{2}{3} \right]$$

$$= 2\pi \frac{60 - 12 - 24 + 120 - 15 - 40}{60}$$

$$= \boxed{\frac{89\pi}{30}}$$

[5]

Question 4 [5 points]: Determine the length of the curve $y = \ln(\sec(x))$ over the interval $0 \le x \le \pi/4$.

$$L = \int_0^{\frac{\pi}{4}} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$$

$$= \int_0^{\frac{\pi}{4}} \sqrt{1 + \left(\frac{\sec(x)\tan(x)}{\sec(x)}\right)^2}\,dx$$

$$= \int_0^{\frac{\pi}{4}} \sqrt{1 + \tan^2(x)}\,dx$$

$$= \int_0^{\frac{\pi}{4}} \sqrt{\sec^2(x)}\,dx$$

$$\rightarrow \int_0^{\frac{\pi}{4}} \sec(x)\,dx$$

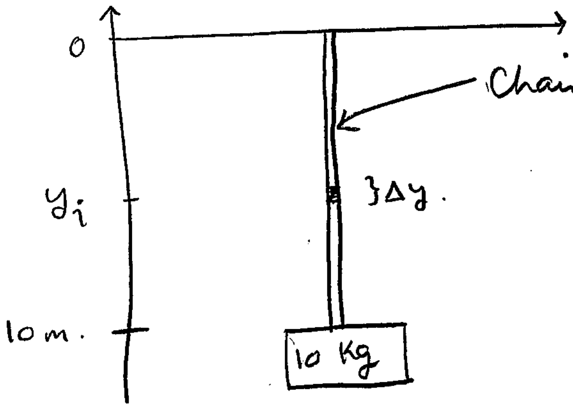
$$= \ln\left|\sec(x) + \tan(x)\right| \Big|_0^{\frac{\pi}{4}}$$

$$= \ln\left|(\sqrt{2}+1)\right| - \underbrace{\cancel{\ln|1+0|}}_{0}$$

$$= \ln(\sqrt{2}+1).$$

[5]

Question 5 [10 points]: A 10 m chain of mass of 2 kg hangs freely over the side of a bridge. At the end of the chain is attaced a 10 kg brick. A person pulls the chain and brick up onto the bridge deck. How much work is done? You may leave your answer in a form which includes the constant g. (Recall: acceleration due to gravity is $g = 9.8$ m/s^2.)



Chain : $\frac{2 \text{ kg}}{10 \text{ m}} = \frac{1}{5} \frac{\text{kg}}{\text{m}}$.

Work to lift chain : segment of length Δy at y_i has weight $\left(\frac{1}{5} \frac{\text{kg}}{\text{m}}\right)(\Delta y \text{ m})\left(g \frac{\text{m}}{\text{s}^2}\right) = \frac{1}{5} g \Delta y$ N

$\therefore$ Work to lift segment to bridge is $\left(\frac{1}{5} g \Delta y \text{ N}\right)(y_i \text{ m}) = \frac{1}{5} g y_i \Delta y$ J

$\therefore$ Total work is $W_{chain} \approx \sum_{i=1}^{n} \frac{1}{5} g y_i \Delta y$.

$\therefore$ $W_{chain} = \int_0^{10} \frac{1}{5} g y \, \Delta y = \frac{g}{5}\left(\frac{y^2}{2}\right)\Big|_0^{10} = 10 g$ J.

Work to lift brick: $W_{brick} = (10 \text{ kg})\left(g \frac{\text{m}}{\text{s}^2}\right)(10 \text{ m}) = 100 g$ J.

$\therefore$ total work $W = W_{chain} + W_{brick}$

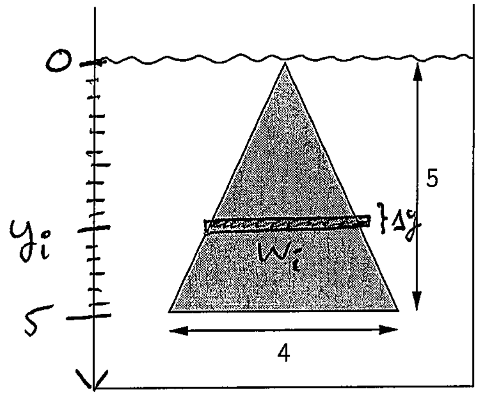
$= 10g + 100g$

$= \boxed{110 g \text{ J}}$

[10]

Question 6 [10 points]:

A metal plate in the shape of an isoceles triangle having base 4 m and height 5 m is submerged in water as shown. Set up BUT DO NOT EVALUATE the integral which gives the hydrostatic force (force due to water pressure) against one side of the plate. Recall that pressure P as a function of depth h is $P(h) = \rho g h$ where ρ is the density of the liquid and g is acceleration due to gravity. Leave the constants ρ and g in your answer— do not convert them to their numerical values.



By similar triangles, strip at y_i has width w_i given by $\frac{w_i}{y_i} = \frac{4}{5}$.

$$\therefore w_i = \frac{4}{5} y_i$$

$\therefore$ area of strip is $\left(\frac{4}{5} y_i\right)(\Delta y)$

$\therefore$ force against strip is $\left(\frac{4}{5} y_i \Delta y\right) P(y_i)$

$$= \frac{4}{5} y_i \Delta y \, \rho g \, y_i$$

$\therefore$ Total force against plate is $\approx \sum_{i=1}^{n} \frac{4}{5} \rho g \, y_i^2 \Delta y$

Letting $n \to \infty$,

$$\boxed{F = \int_0^5 \left(\frac{4}{5}\right) \rho g \, y^2 \, dy.}$$

[10]