Question 1 [10 points]: Determine

$$I = \int \frac{1}{x^2\sqrt{16-x^2}}\,dx$$

Let $x = 4\sin\theta$

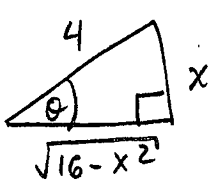
$dx = 4\cos\theta\, d\theta$

$$\therefore\ I = \int \frac{1}{16\sin^2\theta\sqrt{16-16\sin^2\theta}}\, 4\cos\theta\, d\theta$$

$$= \frac{4}{64}\int \frac{1}{\sin^2\theta}\frac{\cos\theta}{\cos\theta}\, d\theta$$

$$= \frac{1}{16}\int \csc^2\theta\, d\theta$$

$$= \frac{-1}{16}\cot\theta + C$$



$$\therefore\ I = \boxed{\frac{-1}{16}\frac{\sqrt{16-x^2}}{x} + C}$$

[10]

Question 2 [10 points]: Determine

$$I = \int \frac{5x^2+3x-2}{x^2(x+2)}\,dx$$

$$\frac{5x^2+3x-2}{x^2(x+2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+2}$$

$$= \frac{Ax(x+2) + B(x+2) + Cx^2}{x^2(x+2)}$$

$$= \frac{(A+C)x^2 + (2A+B)x + 2B}{x^2(x+2)}$$

$$\therefore \quad 2B = -2 \Rightarrow \boxed{B = -1}$$

$$2A+B = 3 \Rightarrow \boxed{A = \frac{3-B}{2} = \frac{4}{2} = 2}$$

$$A+C = 5 \Rightarrow \boxed{C = 5-A = 5-2 = 3}$$

$$\therefore \quad I = \int \frac{2}{x} - \frac{1}{x^2} + \frac{3}{x+2}\,dx$$

$$= \boxed{2\ln|x| + \frac{1}{x} + 3\ln|x+2| + C}$$

[10]

Question 3 [10 points]: Determine the following integrals:

(a) $\frac{1}{2}\int \frac{2x}{\sqrt{9+x^2}}\,dx$ Let $u = 9+x^2$

$du = 2x\,dx$

$\therefore \quad I = \frac{1}{2}\int u^{-1/2}\,du$

$= \frac{1}{\cancel{2}} \cdot \cancel{2}\, u^{1/2} + C$

$= \boxed{\sqrt{9+x^2} + C}$

[5]

(b) $\int \frac{1}{x^2+2x+5}\,dx$

$= \int \frac{1}{(x+1)^2+4}\,dx$ Let $u = x+1$, $du = dx$

$= \int \frac{1}{u^2+2^2}\,du$

$= \frac{1}{2}\arctan\left(\frac{u}{2}\right) + C$

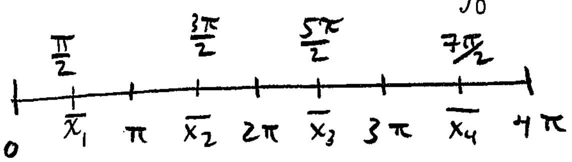
$= \frac{1}{2}\arctan\left(\frac{x+1}{2}\right) + C$

[5]

Question 4 [10 points]:

(a) Use M_4, the midpoint rule on four subintervals to approximate

$$\int_0^{4\pi} e^{\cos(x)} \sin^2(x)\, dx$$



$f(x) = e^{\cos(x)} \sin^2(x)$.

$\Delta x = \pi$

$$M_4 = \Delta x \left[f(\bar{x}_1) + f(\bar{x}_2) + f(\bar{x}_3) + f(\bar{x}_4) \right]$$

$$= \pi \left[e^0 \cdot (1)^2 + e^0 \cdot (-1)^2 + e^0 \cdot (1)^2 + e^0 \cdot (-1)^2 \right]$$

$$= \boxed{4\pi}$$

[5]

(b) Suppose you are using the trapezoid rule on 4 subintervals to approximate $\int_0^4 \sin(x/2)\, dx$. Determine an error bound $|E_{T_4}|$ on the resulting approximation. (Recall: the error in using the trapezoid rule to approximate $\int_a^b f(x)\, dx$ using n subintervals is at most $\frac{K(b-a)^3}{12n^2}$ where $|f''(x)| \leq K$ on [a,b].)

$$f(x) = \sin\left(\frac{x}{2}\right)$$

$$f'(x) = \frac{1}{2}\cos\left(\frac{x}{2}\right)$$

$$f''(x) = \frac{-1}{4}\sin\left(\frac{x}{2}\right).$$

$$|f''(x)| = \left| \frac{-1}{4}\sin\left(\frac{x}{2}\right) \right| \leq \frac{1}{4} \quad \text{on} \quad [0,4].$$

$$\therefore \quad K = \frac{1}{4}, \; [a,b] = [0,4], \; n = 4$$

$$\therefore \quad |E_{T_4}| \leq \frac{\cancel{\frac{1}{4}}\,\cancel{(4-0)^3}^{\,2}}{12 \cdot \cancel{n^2}} = \boxed{\frac{1}{12}}$$

[5]

Question 5 [10 points]:

(a) Evaluate the improper integral $\int_0^\infty x^2 e^{-x^3}\, dx$ making proper use of any required limits.

$$\int_0^\infty x^2 e^{-x^3} dx$$

$$= \lim_{b\to\infty} \int_0^b x^2 e^{-x^3} dx \quad \left\} \begin{array}{l} u = -x^3 \\ du = -3x^2\, dx \end{array} \right.$$

$$= \lim_{b\to\infty} -\frac{1}{3}\left[e^{-x^3}\right]_0^b$$

$$= \lim_{b\to\infty} -\frac{1}{3}\left[e^{-b^3} - e^0\right]$$

$$= \boxed{\frac{1}{3}}$$

[5]

(b) Use the comparison theorem to determine whether $\int_1^\infty \frac{\cos^2(x)}{1+x^2}\, dx$ is convergent or divergent.

on $[1,\infty)$ $\quad 0 \leq \frac{\cos^2(x)}{1+x^2} \leq \frac{1}{x^2}$.

Since $\int_1^\infty \frac{1}{x^2} dx$ converges (p-integral, $p = 2 > 1$),

So does $\int_1^\infty \frac{\cos^2(x)}{1+x^2} dx$ by the comparison theorem.

[5]