Question 1 [10 points]:

(a) Determine the value of b so that the average value of $f(x) = x^2$ over $[0, b]$ is same as that of $g(x) = x^3$ over $[0, 2b]$.

$$\frac{1}{b}\int_0^b x^2\,dx = \frac{1}{2b}\int_0^{2b} x^3\,dx$$

$$\frac{1}{b}\left[\frac{x^3}{3}\right]_0^b = \frac{1}{2b}\left[\frac{x^4}{4}\right]_0^{2b}$$

$$\frac{b^3}{3b} = \frac{16b^4}{8b}$$

$$\frac{\cancel{b^2}}{3} = 2\cancel{b^3} \implies \boxed{b = \frac{1}{6}}$$

[3]

(b) For this question use the equation

$$6 + \int_a^x \frac{f(t)}{t^2}\,dt = 2\sqrt{x} \quad \Big\} (*)$$

(i) Determine the value of the constant a .

Set $x = a$ in $(*)$: $6 + 0 = 2\sqrt{a}$

$$\Rightarrow \sqrt{a} = 3$$

$$\Rightarrow \boxed{a = 9}$$

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[2]

(ii) Determine the function f .

$$\frac{d}{dx}\left[6 + \int_a^x \frac{f(t)}{t^2}\,dt\right] = \frac{d}{dx}\left[2\sqrt{x}\right]$$

$$\frac{f(x)}{x^2} = \frac{1}{\sqrt{x}}$$

$$\therefore f(x) = \frac{x^2}{\sqrt{x}} = \boxed{x^{3/2}}$$

[2]

(c) Determine $f'(0)$ if $f(x) = \int_0^{(1+x)^2} t^2 e^t\,dt$.

$$f'(x) = \left((1+x)^2\right)^2 e^{(1+x)^2}\, 2(1+x)$$

$$f'(0) = 1 \cdot e^1 \cdot 2 \cdot 1 = \boxed{2e}$$

[3]

Question 2 [10 points]:

(a) Determine $\int_1^2 \frac{t^5 - 2t}{t^3}\, dt$.

$$= \int_1^2 t^2 - 2t^{-2}\, dt$$

$$= \left[\frac{t^3}{3} + \frac{2}{t}\right]_1^2$$

$$= \left(\frac{8}{3} + 1\right) - \left(\frac{1}{3} + 2\right) = \boxed{\frac{4}{3}}$$

[3]

(b) Determine $\int 4\sec^2(x) + \frac{\pi}{x^2}\, dx$.

$$= \boxed{4\tan(x) - \frac{\pi}{x} + C}$$

[3]

(c) A tree's height increases at a rate of $h'(t) = \frac{3}{\sqrt{1+t}} + \frac{2}{1+t}$ meters per year where $t = 0$ corresponds to the present. What is the increase in height during the first three years of growth?

$$h(3) - h(0) = \int_0^3 \frac{3}{\sqrt{1+t}} + \frac{2}{1+t}\, dt$$

$$= \left[6\sqrt{1+t} + 2\ln|1+t|\right]_0^3$$

$$= \left(12 + 2\ln(4)\right) - \left(6 + 2\underbrace{\ln(1)}_{0}\right)$$

$$= \boxed{6 + 2\ln(4) \text{ m}}$$

[4]

Question 3 [10 points]: Determine the following integrals:

(a) $\int \frac{1}{x^2}\sqrt{3-\frac{1}{x}}\,dx$ $\quad \left.\begin{array}{l} u = 3-\frac{1}{x} \\ du = \frac{1}{x^2}\,dx \end{array}\right\}$

$= \int u^{1/2}\,du$

$= \frac{2}{3}u^{3/2} + C$

$= \boxed{\frac{2}{3}\left(3-\frac{1}{x}\right)^{\frac{3}{2}} + C}$

[3]

(b) $\int x\cos(2x^2)\,dx$ $\quad \left.\begin{array}{l} u = 2x^2 \\ du = 4x\,dx \end{array}\right\}$

$= \frac{1}{4}\int \cos(u)\,du$

$= \frac{1}{4}\sin(u) + C$

$= \boxed{\frac{1}{4}\sin(2x^2) + C}$

[3]

(c) $\int \sec^3(x)\tan(x)\,dx$

$= \int \sec^2(x)\sec(x)\tan(x)\,dx$ $\quad \left.\begin{array}{l} u = \sec(x) \\ du = \sec(x)\tan(x)\,dx \end{array}\right\}$

$= \int u^2\,du$

$= \frac{u^3}{3} + C$

$= \boxed{\frac{\sec^3(x)}{3} + C}$

[4]

Question 4 [10 points]: Determine the following integrals:

(a) $\int_0^{\pi} x \sin(x)\, dx$ $\quad \left\} \begin{array}{ll} u = x & dv = \sin(x)dx \\ du = dx & v = -\cos(x) \end{array} \right.$

$$= \Big[-x\cos(x)\Big]_0^{\pi} - \int_0^{\pi} -\cos(x)\, dx$$

$$= -\pi\cos(\pi) + 0\cdot\cos(0) + \Big[\sin(x)\Big]_0^{\pi}$$

$$= \pi + \cancel{\sin(\pi)}^{0} - \cancel{\sin(0)}^{0}$$

$$= \boxed{\pi}$$

[5]

(b) $\int (x^2+1)e^x\, dx$ $\quad \left\} \begin{array}{ll} u = x^2+1 & dv = e^x dx \\ du = 2x\, dx & v = e^x \end{array} \right.$

$$= (x^2+1)e^x - 2\underbrace{\int x e^x\, dx}_{\substack{u = x \quad dv = e^x dx \\ du = dx \quad v = e^x}}$$

$$= (x^2+1)e^x - 2\left[x e^x - \int e^x dx\right]$$

$$= \boxed{(x^2+1)e^x - 2x e^x + 2e^x + C}$$

[5]

Question 5 [10 points]: Determine the following integrals:

(a) $\int \sin^3(x)\cos^2(x)\,dx$

$= \int \sin^2(x)\cos^2(x)\ \sin(x)\,dx$

$= -\int (1-\cos^2(x))\cos^2(x)(-\sin(x))\,dx \quad \left\{ \begin{array}{l} u = \cos(x) \\ du = -\sin(x)\,dx \end{array} \right.$

$= -\int (1-u^2)u^2\,du$

$= -\int u^2 - u^4\,du$

$= -\left[\frac{u^3}{3} - \frac{u^5}{5}\right] + C$

$= \boxed{-\frac{\cos^3(x)}{3} + \frac{\cos^5(x)}{5} + C}$

[5]

(b) $\int \tan^3(x)\,dx$

$= \int \tan(x)\tan^2(x)\,dx$

$= \int \tan(x)(\sec^2(x)-1)\,dx$

$= \underbrace{\int \tan(x)\sec^2(x)\,dx}_{\substack{u=\tan(x) \\ du = \sec^2(x)dx}} - \underbrace{\int \frac{\sin x}{\cos x}\,dx}_{\substack{w=\cos(x) \\ dw = -\sin(x)dx}}$

$= \int u\,du + \int \frac{1}{w}\,dw$

$= \frac{u^2}{2} + \ln|w| + C = \boxed{\frac{\tan^2(x)}{2} + \ln|\cos(x)| + C}$

[5]