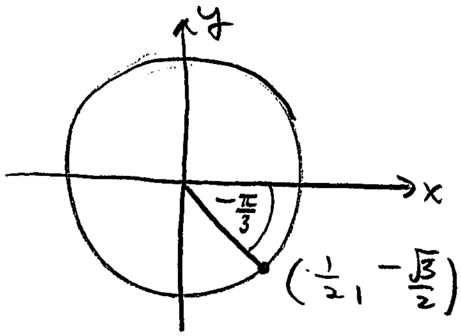
Question 1 [10 points]:

(a) Determine $\arcsin\left(-\sqrt{3}/2\right)$



$$\arcsin\left(-\frac{\sqrt{3}}{2}\right) = \boxed{-\frac{\pi}{3}}$$

[2]

(b) Determine $\sec\left(\cos^{-1}(1/2)\right)$.

$$\sec\left(\cos^{-1}\left(\tfrac{1}{2}\right)\right) = \frac{1}{\cos\left(\cos^{-1}\left(\frac{1}{2}\right)\right)} = \frac{1}{\left(\frac{1}{2}\right)} = \boxed{2.}$$

[2]

(c) Let $f(x) = \arctan\left(\sqrt{x^2-1}\right)$. Calculate and simplify $f''(2)$.

$$f'(x) = \frac{1}{1+\left(\sqrt{x^2-1}\right)^2} \cdot \frac{1}{\cancel{2}}(x^2-1)^{-\frac{1}{2}}(\cancel{2}x) = \frac{1}{x\sqrt{x^2-1}} = x^{-1}(x^2-1)^{-\frac{1}{2}}$$

$$f''(x) = -x^{-2}(x^2-1)^{-\frac{1}{2}} + \cancel{x^{-1}}\left(-\frac{1}{\cancel{2}}\right)(x^2-1)^{-\frac{3}{2}}(\cancel{2x})$$

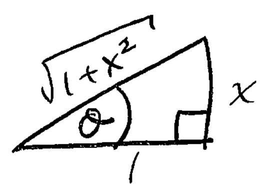
$$f''(2) = -\frac{1}{4\sqrt{3}} - \frac{1}{3\sqrt{3}} = \boxed{\frac{-7}{12\sqrt{3}}.} \text{ or } \boxed{\frac{-7\sqrt{3}}{36}}$$

[3]

(d) Express $\csc\left(\tan^{-1}(x)\right)$ in a simplified form that does not contain any trigonometric or inverse trigonometric functions.

Let $\theta = \tan^{-1}(x)$

$\Rightarrow \tan\theta = \frac{x}{1}$:



$$\therefore \csc\left(\tan^{-1}(x)\right) = \csc(\theta) = \boxed{\frac{\sqrt{1+x^2}}{x}}$$

[3]

Question 2 [10 points]:

(a) Let $f(x) = 2\cosh(\ln(x))$. Find $f'(x)$ and simplify your answer so that is does not contain $\ln(x)$.

$$f(x) = 2\cosh(\ln(x)) = 2\left[\frac{e^{\ln(x)} + e^{-\ln(x)}}{2}\right] = e^{\ln(x)} + e^{\ln\left(\frac{1}{x}\right)} = x + \frac{1}{x}$$

$$\therefore f'(x) = \boxed{1 - \frac{1}{x^2}}$$

[3]

(b) Determine $\lim_{x\to\infty}(\cosh x - \sinh x)$.

$$\cosh x - \sinh x = \frac{e^x + e^{-x}}{2} - \frac{e^x - e^{-x}}{2} = e^{-x}$$

$$\therefore \lim_{x\to\infty}(\cosh x - \sinh x) = \lim_{x\to\infty} e^{-x} = \boxed{0}$$

[3]

(c) Determine the x-coordinate of the point on the graph of $y = \cosh(x)$ at which the tangent line has slope 1.

Solve $y' = 1$ for $x \Rightarrow \sinh(x) = 1$

$$\frac{e^x - e^{-x}}{2} = 1 \Rightarrow e^x - e^{-x} = 2 \Rightarrow e^{2x} - 2e^x - 1 = 0$$

$$\therefore e^x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(-1)}}{2}$$

$$= \frac{2 \pm 2\sqrt{2}}{2}$$

$$= 1 \pm \sqrt{2}.$$

$\therefore x = \ln(1+\sqrt{2})$, $\underbrace{x = \cancel{\ln(1-\sqrt{2})}}_{\text{not defined.}}$

[4]

Question 3 [10 points]: Find the following limits:

(a) $\lim_{t\to 0} \frac{\tan(5t)}{2t} \sim "\frac{0}{0}"$

$$\overset{H}{=} \lim_{t\to 0} \frac{\sec^2(5t)\,5}{2} = \boxed{\frac{5}{2}}$$

[2]

(b) $\lim_{x\to 0} \frac{\sin(x) - x}{x^3} \sim "\frac{0}{0}"$

$$\overset{H}{=} \lim_{x\to 0} \frac{\cos(x) - 1}{3x^2} \sim "\frac{0}{0}"$$

$$\overset{H}{=} \lim_{x\to 0} \frac{-\sin(x)}{6x} \sim "\frac{0}{0}"$$

$$\overset{H}{=} \lim_{x\to 0} \frac{-\cos(x)}{6} = \boxed{\frac{-1}{6}}$$

[3]

(c) $\lim_{x\to 0^+} x(\ln(x))^2 \sim 0 \cdot \infty$.

$$= \lim_{x\to 0^+} \frac{(\ln(x))^2}{\left(\frac{1}{x}\right)} \sim "\frac{\infty}{\infty}"$$

$$\overset{H}{=} \lim_{x\to 0^+} \frac{2\ln(x)\left(\frac{1}{x}\right)}{-\frac{1}{x^2}}$$

$$= \lim_{x\to 0^+} -2\,\frac{\ln(x)}{\frac{1}{x}} \sim "\frac{\infty}{\infty}"$$

$$\overset{H}{=} \lim_{x\to 0^+} -2\,\frac{\left(\frac{1}{x}\right)}{\frac{-1}{x^2}}$$

$$= \lim_{x\to 0^+} 2x$$

$$= \boxed{0}.$$

[5].

Question 4 [10 points]:

(a) Determine the most general antiderivative of $f(x) = \frac{4}{x} - \frac{5}{\sqrt{1-x^2}}$.

$$\boxed{F(x) = 4\ln|x| - 5\arcsin(x) + C}$$

or $$\boxed{F(x) = 4\ln|x| + 5\arccos(x) + C}$$

[3]

(b) Determine the most general antiderivative of $g(t) = \frac{t^3 + \sqrt{t} - 4}{t^3}$.

$$g(t) = 1 + t^{-5/2} - 4t^{-3}$$

$$\boxed{G(t) = t - \frac{2}{3}t^{-\frac{3}{2}} + 2t^{-2} + C}$$

[3]

(c) Determine $f(x)$ if $f''(x) = \pi\cos(x) - 2e^x$ and $f(0) = -\pi$, $f'(0) = 0$.

$$f'(x) = \pi\sin(x) - 2e^x + C_1$$

$$f'(0) = 0 \Rightarrow \pi\,\cancel{\sin(0)}^{0} - 2\cancel{e^0}^{1} + C_1 = 0$$

$$\Rightarrow C_1 = 2.$$

$$\therefore f'(x) = \pi\sin(x) - 2e^x + 2,$$

$$f(x) = -\pi\cos(x) - 2e^x + 2x + C_2$$

$$f(0) = -\pi \Rightarrow -\pi\,\cancel{\cos(0)}^{1} - 2\cancel{e^0}^{1} + \cancel{2\cdot 0}^{0} + C_2 = -\pi$$

$$-\pi - 2 + C_2 = -\pi$$

$$\Rightarrow C_2 = 2,$$

[4]

$$\therefore \boxed{f(x) = -\pi\cos(x) - 2e^x + 2x + 2}$$

Question 5 [10 points]: Use the definition of the definite integral in the form

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

to evaluate

$$\int_0^3 (x^2 - 3x)\,dx$$

Carefully set up the Riemann sum and clearly show the steps of your simplification.

$$\Delta x = \frac{b-a}{n} = \frac{3-0}{n} = \frac{3}{n}$$

$$x_i = a + i\Delta x = 0 + i\left(\frac{3}{n}\right) = \frac{3i}{n}, \quad i = 1, 2, \dots, n,$$

$$f(x_i) = x_i^2 - 3x_i = \frac{9i^2}{n^2} - \frac{9i}{n}.$$

$$\therefore \int_0^3 (x^2 - 3x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left[\frac{9i^2}{n^2} - \frac{9i}{n}\right]\left(\frac{3}{n}\right)$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left[\frac{27i^2}{n^3} - \frac{27i}{n^2}\right]$$

$$= \lim_{n\to\infty} \left[\frac{27}{n^3}\left(\sum_{i=1}^{n} i^2\right) - \frac{27}{n^2}\left(\sum_{i=1}^{n} i\right)\right]$$

$$= \lim_{n\to\infty} \left[\frac{27}{n^3}\,\frac{n(n+1)(2n+1)}{6} - \frac{27}{n^2}\,\frac{n(n+1)}{2}\right]$$

$$= \lim_{n\to\infty} \left[\frac{27}{6}\cdot\frac{n}{n}\cdot\frac{n+1}{n}\cdot\frac{2n+1}{n} - \frac{27}{2}\cdot\frac{n}{n}\cdot\frac{n+1}{n}\right]$$

$$= \left(\frac{27}{6}\right)(1)(1)(2) - \left(\frac{27}{2}\right)(1)(1)$$

$$= \boxed{\frac{-9}{2}}$$

[10]