Question 1 [10 points]:

(a) Determine $f'(0)$ if $f(x) = \int_0^{(1+x)^2} t^2 e^t \, dt$.

$$f'(x) = ((1+x)^2)^2 e^{(1+x)^2} \cdot 2(1+x)$$

$$= (1+x)^4 e^{(1+x)^2} \cdot 2(1+x)$$

$$f'(0) = 1 \cdot e^1 \cdot 2 \cdot 1 = \boxed{2e}$$

[3]

(b) For this question use the equation

$$\left. 6 + \int_a^x \frac{f(t)}{t^2} \, dt = 2\sqrt{x} \quad \right\} \quad (*)$$

(i) Determine the value of the constant a .

Let $x = a$ in $(*)$:

$$6 + 0 = 2\sqrt{a}$$

$$\Rightarrow \sqrt{a} = 3$$

$$\Rightarrow \boxed{a = 9}.$$

This is HW Question 5.4 #31

[2]

(ii) Determine the function f .

$$\frac{d}{dx}\left[6 + \int_a^x \frac{f(t)}{t^2} dt\right] = \frac{d}{dx}\left[2\sqrt{x}\right]$$

$$\Rightarrow \frac{f(x)}{x^2} = \frac{1}{\sqrt{x}}$$

$$\Rightarrow f(x) = \frac{x^2}{\sqrt{x}} = \boxed{x^{3/2}}$$

[2]

(c) Determine the value of b so that the average value of $f(x) = x^2$ over $[0, b]$ is same as that of $g(x) = x^3$ over $[0, 2b]$.

$$\frac{1}{b}\int_0^b x^2 \, dx = \frac{1}{2b}\int_0^{2b} x^3 \, dx$$

$$\Rightarrow \frac{1}{b}\left[\frac{x^3}{3}\right]_0^b = \frac{1}{2b}\left[\frac{x^4}{4}\right]_0^{2b}$$

$$\Rightarrow \frac{1}{b} \frac{b^3}{3} = \frac{16 b^4}{8b}$$

$$\Rightarrow \frac{\cancel{b^2}}{3} = 2 \cancel{b^3} \Longrightarrow \boxed{b = \frac{1}{6}}$$

[3]

Question 2 [10 points]:

(a) Determine $\int 4\sec^2(x) + \frac{\pi}{x^2}\, dx$.

$$= \boxed{4\tan(x) - \frac{\pi}{x} + C}$$

[3]

(b) Determine $\int_1^2 \frac{t^5 - 2t}{t^3}\, dt$.

$$= \int_1^2 t^2 - 2t^{-2}\, dt$$

$$= \left[\frac{t^3}{3} + \frac{2}{t}\right]_1^2$$

$$= \left(\frac{8}{3} + 1\right) - \left(\frac{1}{3} + 2\right) = \frac{8+3-\cancel{1+6}}{3} = \boxed{\frac{4}{3}}$$

[3]

(c) A tree's height increases at a rate of $h'(t) = \frac{2}{\sqrt{1+t}} + \frac{3}{1+t}$ meters per year where $t = 0$ corresponds to the present. What is the increase in height during the first three years of growth?

$$h(3) - h(0) = \int_0^3 \frac{2}{\sqrt{1+t}} + \frac{3}{1+t}\, dt$$

$$= \left[4\sqrt{1+t} + 3\ln|1+t|\right]_0^3$$

$$= \left(4\sqrt{4} + 3\ln(4)\right) - \left(4\sqrt{1} + 3\underbrace{\ln(1)}_{0}\right)$$

$$= \boxed{4 + 3\ln(4) \quad m}.$$

[4]

Question 3 [10 points]: Determine the following integrals:

(a) $\int x \sin(2x^2)\, dx = I$

let $u = 2x^2$

$du = 4x\, dx$

$\therefore I = \frac{1}{4}\int \sin(u)\, du$

$= \frac{-1}{4}\cos(u) + C$

$= \boxed{-\frac{1}{4}\cos(2x^2) + C}$

[3]

(b) $\int \frac{1}{x^2}\sqrt{2 - \frac{1}{x}}\, dx = I$

let $u = 2 - \frac{1}{x}$

$du = \frac{1}{x^2}\, dx$

$\therefore I = \int \sqrt{u}\, du$

$= \frac{2}{3}u^{3/2} + C$

$= \boxed{\frac{2}{3}\left(2 - \frac{1}{x}\right)^{3/2} + C}$

[3]

(c) $I = \int \sec^3(x)\tan(x)\, dx$

$= \int \sec^2(x) \cdot \sec(x)\tan(x)\, dx$

$u = \sec(x)$

$du = \sec(x)\tan(x)\, dx$

$\therefore I = \int u^2\, du$

$= \frac{1}{3}u^3 + C$

$= \boxed{\frac{1}{3}\sec^3(x) + C}$

[4]

Question 4 [10 points]: Determine the following integrals:

(a) $\int_0^\pi x\cos(x)\,dx$ $\quad u = x \quad dv = \cos(x)dx$

$du = dx \quad v = \sin(x)$

$= uv - \int v\,du$

$= \left[x\sin(x)\right]_0^\pi - \int_0^\pi \sin(x)dx$

$= \pi\cdot\underbrace{\sin(\pi)}_{0} - 0\cdot\underbrace{\sin(0)}_{0} + \left[\cos(x)\right]_0^\pi$

$= \cos(\pi) - \cos(0)$

$= -1 - 1$

$= \boxed{-2}$

[5]

(b) $\int (x^2-1)e^x\,dx$ $\quad u = x^2-1 \quad dv = e^x dx$

$du = 2x\,dx \quad v = e^x$

$= uv - \int v\,du$

$= (x^2-1)e^x - 2\underbrace{\int xe^x dx}$

$u = x \quad dv = e^x dx$

$du = dx \quad v = e^x$

$= (x^2-1)e^x - 2\left[xe^x - \int e^x dx\right]$

$= \boxed{(x^2-1)e^x - 2xe^x + 2e^x + C}$

[5]

Question 5 [10 points]: Determine the following integrals:

(a) $\int \sin^2(x)\cos^3(x)\,dx$

$$= \int \sin^2(x)\cos^2(x)\cos(x)\,dx$$

$$= \int \sin^2(x)\left(1-\sin^2(x)\right)\cos(x)\,dx \quad \left\} \begin{array}{l} u = \sin(x) \\ du = \cos(x)dx \end{array} \right.$$

$$= \int u^2(1-u^2)\,du$$

$$= \int (u^2 - u^4)\,du$$

$$= \frac{u^3}{3} - \frac{u^5}{5} + C$$

$$= \boxed{\frac{\sin^3(x)}{3} - \frac{\sin^5(x)}{5} + C}$$

[5]

(b) $\int \tan^3(x)\,dx$

$$= \int \tan(x)\tan^2(x)\,dx$$

$$= \int \tan(x)\left(\sec^2(x) - 1\right)dx$$

$$= \underbrace{\int \tan(x)\sec^2(x)\,dx}_{\substack{u=\tan(x) \\ du = \sec^2(x)dx}} - \underbrace{\int \frac{\sin x}{\cos x}\,dx}_{\substack{w = \cos(x) \\ dw = -\sin(x)dx}}$$

$$= \int u\,du + \int \frac{1}{w}\,dw$$

$$= \frac{u^2}{2} + \ln|w| + C$$

$$= \boxed{\frac{\tan^2(x)}{2} + \ln|\cos(x)| + C}$$

[5]