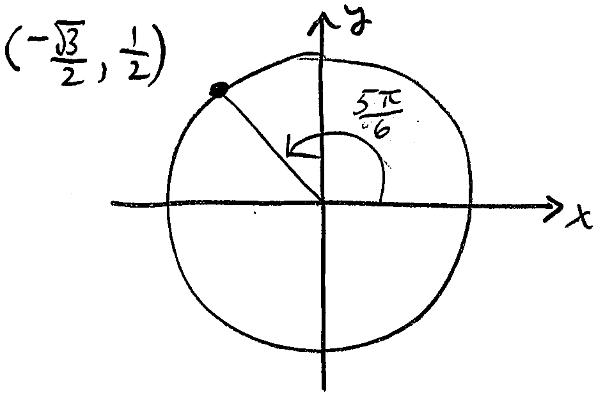
Question 1 [10 points]:

(a) Determine $\csc(\sin^{-1}(1/2))$.

$$\csc\left(\sin^{-1}\left(\tfrac{1}{2}\right)\right) = \frac{1}{\sin\left(\sin^{-1}\left(\frac{1}{2}\right)\right)} = \frac{1}{\left(\frac{1}{2}\right)} = \boxed{2}$$

[2]

(b) Determine $\arccos(-\sqrt{3}/2)$



$$\arccos\left(-\frac{\sqrt{3}}{2}\right) = \boxed{\frac{5\pi}{6}}$$

[2]

(c) Let $f(x) = \arctan(\sqrt{x^2-1})$. Calculate and simplify $f''(2)$.

$$f'(x) = \frac{1}{1+\left(\sqrt{x^2-1}\right)^2} \cdot \cancel{\tfrac{1}{2}}(x^2-1)^{-\frac{1}{2}}(\cancel{2}x) = \frac{1}{x\sqrt{x^2-1}} = x^{-1}(x^2-1)^{-\frac{1}{2}}$$

$$f''(x) = -x^{-2}(x^2-1)^{-\frac{1}{2}} + \cancel{x^{-1}}\left(\frac{-1}{\cancel{2}}\right)(x^2-1)^{-3/2}(\cancel{2x})$$

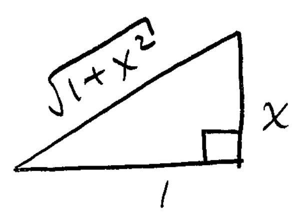
$$f''(2) = -\frac{1}{4\sqrt{3}} - \frac{1}{3\sqrt{3}} = \boxed{-\frac{7}{12\sqrt{3}}} \text{ or } \boxed{\frac{-7\sqrt{3}}{36}}$$

[3]

(d) Express $\sec(\tan^{-1}(x))$ in a simplified form that does not contain any trigonometric or inverse trigonometric functions.

Let $\theta = \tan^{-1}(x)$

$\Rightarrow \tan\theta = \frac{x}{1}$:



$$\therefore \sec\left(\tan^{-1}(x)\right) = \sec(\theta) = \boxed{\sqrt{1+x^2}}$$

[3]

Question 2 [10 points]:

(a) Determine $\lim_{x\to-\infty}(\sinh x + \cosh x)$. $= \lim_{x\to-\infty}\left(\frac{e^{x}-e^{-x}}{2} + \frac{e^{x}+e^{-x}}{2}\right)$

$$= \lim_{x\to-\infty} e^{x}$$

$$= \boxed{0}$$

[3]

(b) Let $f(x) = 2\sinh(\ln(x))$. Find $f'(x)$ and simplify your answer so that is does not contain $\ln(x)$.

$$f(x) = 2\left[\frac{e^{\ln(x)} - e^{-\ln(x)}}{2}\right] = e^{\ln(x)} - e^{\ln(\frac{1}{x})} = x - \frac{1}{x}$$

$$\therefore\ f'(x) = \boxed{1 + \frac{1}{x^2}}$$

[3]

(c) Determine the x-coordinate of the point on the graph of $y = \cosh(x)$ at which the tangent line has slope -1.

Solve $y' = -1$ for x.

$$\therefore\ \sinh(x) = -1 \Rightarrow \frac{e^{x}-e^{-x}}{2} = -1$$

$$\Rightarrow e^{x}-e^{-x} = -2$$

$$e^{2x} + 2e^{x} - 1 = 0$$

$$\Rightarrow e^{x} = \frac{-2 \pm \sqrt{2^2 - 4(1)(-1)}}{2} = \frac{-2 \pm 2\sqrt{2}}{2}$$

$$= -1+\sqrt{2},\ -1-\sqrt{2}$$

$$\therefore\ x = \ln(-1+\sqrt{2}),\ \underbrace{\cancel{\ln(-1-\sqrt{2})}}_{\text{not defined.}}$$

$$\boxed{\therefore\ x = \ln(-1+\sqrt{2})}$$

[4]

Question 3 [10 points]: Find the following limits:

(a) $\lim_{t\to 0} \frac{\tan(2t)}{5t}$ $\sim$ "$\frac{0}{0}$"

$\stackrel{H}{=} \lim_{t\to 0} \frac{\sec^2(2t)\cdot 2}{5}$

$= \boxed{\frac{2}{5}}$

[2]

(b) $\lim_{x\to 0} \frac{x-\sin(x)}{x^3}$ $\sim$ "$\frac{0}{0}$"

$\stackrel{H}{=} \lim_{x\to 0} \frac{1-\cos(x)}{3x^2}$ $\sim$ "$\frac{0}{0}$"

$\stackrel{H}{=} \lim_{x\to 0} \frac{\sin(x)}{6x}$ $\sim$ "$\frac{0}{0}$"

$\stackrel{H}{=} \lim_{x\to 0} \frac{\cos(x)}{6} = \boxed{\frac{1}{6}}$

[3]

(c) $\lim_{x\to 0^+} x(\ln(x))^2$ $\sim 0\cdot\infty$

$= \lim_{x\to 0^+} \frac{(\ln(x))^2}{\frac{1}{x}}$ $\sim$ "$\frac{\infty}{\infty}$"

$\stackrel{H}{=} \lim_{x\to 0^+} \frac{2\ln(x)\cdot\frac{1}{x}}{\frac{-1}{x^2}}$

$= \lim_{x\to 0^+} -2\frac{\ln(x)}{\left(\frac{1}{x}\right)}$ $\sim$ "$\frac{+\infty}{\infty}$"

$\stackrel{H}{=} \lim_{x\to 0^+} \frac{-2\left(\frac{1}{x}\right)}{-\frac{1}{x^2}}$

$= \lim_{x\to 0^+} 2x$

$= \boxed{0}$

[5]

Question 4 [10 points]:

(a) Determine the most general antiderivative of $g(t) = \dfrac{t^3 - \sqrt{t} + 7}{t^3}$. $= 1 - t^{-\frac{5}{2}} + 7t^{-3}$

$$\boxed{G(t) = t + \tfrac{2}{3} t^{-\frac{3}{2}} - \tfrac{7}{2} t^{-2} + C}$$

[3]

(b) Determine the most general antiderivative of $f(x) = \dfrac{4}{\sqrt{1-x^2}} - \dfrac{5}{x}$.

$$\boxed{F(x) = 4\arcsin(x) - 5\ln|x| + C}$$

$\underline{\underline{\text{or}}}$

$$\boxed{F(x) = -4\arccos(x) - 5\ln|x| + C}$$

[3]

(c) Determine $f(x)$ if $f''(x) = \pi \sin(x) - 2e^x$ and $f(0) = 0$, $f'(0) = -\pi$.

$f'(x) = -\pi\cos(x) - 2e^x + C_1$.

$f'(0) = -\pi \Rightarrow -\pi - 2 + C_1 = -\pi \Rightarrow C_1 = 2$.

$\therefore f'(x) = -\pi\cos(x) - 2e^x + 2$.

$\therefore f(x) = -\pi\sin(x) - 2e^x + 2x + C_2$.

$f(0) = 0 \Rightarrow -2 + C_2 = 0 \Rightarrow C_2 = 2$.

$$\boxed{\therefore f(x) = -\pi\sin(x) - 2e^x + 2x + 2}$$

[4]

Question 5 [10 points]: Use the definition of the definite integral in the form

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

to evaluate

$$\int_0^3 (x - 3x^2)\,dx$$

Carefully set up the Riemann sum and clearly show the steps of your simplification.

$$\Delta x = \frac{b-a}{n} = \frac{3-0}{n} = \frac{3}{n}$$

$$x_i = a + i\Delta x = 0 + i\left(\frac{3}{n}\right) = \frac{3i}{n}, \quad i = 1, 2, \ldots, n,$$

$$f(x_i) = x_i - 3x_i^2 = \frac{3i}{n} - 3\left(\frac{9i^2}{n^2}\right) = \frac{3i}{n} - \frac{27i^2}{n^2},$$

$$\int_0^3 f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{3i}{n} - 27\frac{i^2}{n^2}\right)\left(\frac{3}{n}\right)$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{9i}{n^2} - 81\frac{i^2}{n^3}\right)$$

$$= \lim_{n\to\infty} \left[\frac{9}{n^2}\left(\sum_{i=1}^{n} i\right) - \frac{81}{n^3}\left(\sum_{i=1}^{n} i^2\right)\right]$$

$$= \lim_{n\to\infty} \left[\frac{9}{n^2}\,\frac{n(n+1)}{2} - \frac{81}{n^3}\,\frac{n(n+1)(2n+1)}{6}\right]$$

$$= \lim_{n\to\infty} \left[\frac{9}{2}\cdot\frac{n}{n}\cdot\frac{n+1}{n} - \frac{81}{6}\cdot\frac{n}{n}\cdot\frac{n+1}{n}\cdot\frac{2n+1}{n}\right]$$

$$= \left(\frac{9}{2}\right)(1)(1) - \left(\frac{81}{6}\right)(1)(1)(2)$$

$$= \boxed{-\frac{45}{2}}$$

[10]