Question 1 [10 points]:

(a) A system of three equations in the variables x, y and z has been partially reduced using Gaussian elimination, resulting in the following matrix:

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 2 \\ 0 & 1 & 2 & 3 \\ 0 & -2 & -4 & -9 \end{array}\right]$$

Complete the matrix reduction and state the solution to the system. If there is no solution, state "no solution".

$R_2 = 2r_2 + r_1$:

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & -3 \end{array}\right]$$

no solution.

[3]

(b) A system of three equations in the variables x and y has been partially reduced using Gaussian elimination, resulting in the following matrix:

$$\left[\begin{array}{cc|c} 1 & -2 & -3 \\ 0 & -3 & -6 \\ 0 & 1 & 2 \end{array}\right]$$

Complete the matrix reduction and state the solution to the system. If there is no solution, state "no solution".

$R_2 = (-\frac{1}{3})r_2$: $\left[\begin{array}{cc|c} 1 & -2 & -3 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{array}\right] \to$ $R_3 = (-1)r_2 + r_3$: $R_1 = 2r_2 + r_1$: $\left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right]$

$\boxed{\therefore\ x = 1,\ y = 2}$

[2]

(c) Solve the following system of equations using Gaussian (or Gauss-Jordan) elimination:

$$2x - 3y + 4z = 7$$
$$x - 2y + 3z = 2$$

$$\left[\begin{array}{ccc|c} 2 & -3 & 4 & 7 \\ 1 & -2 & 3 & 2 \end{array}\right]$$

$r_1 \leftrightarrow r_2$:

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 2 \\ 2 & -3 & 4 & 7 \end{array}\right]$$

$R_2 = (-2)r_1 + r_2$:

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 2 \\ 0 & 1 & -2 & 3 \end{array}\right]$$

$R_1 = 2r_2 + r_1$:

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 8 \\ 0 & 1 & -2 & 3 \end{array}\right].$$

$\therefore$ $y = 3 + 2z$, $x = 8 + z$ } where z is any real number

or $(8+z, 3+2z, z)$

[5]

Question 2 [10 points]: For this problem use the following matrices:

$$\mathbf{A} = \begin{bmatrix} -1 & 4 & 1 \\ 2 & 4 & -3 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 2 & -3 \\ 1 & 1 \\ -3 & -2 \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 0 & 5 \end{bmatrix} \quad \mathbf{D} = \begin{bmatrix} 3 & -2 \\ 1 & 4 \end{bmatrix}$$

Calculate the following, if possible. If an operation is not defined, state "not defined":

(a) $\mathbf{A} + 6\mathbf{C}$ $= \begin{bmatrix} -1 & 4 & 1 \\ 2 & 4 & -3 \end{bmatrix} + 6 \begin{bmatrix} 1 & -1 & 1 \\ 2 & 0 & 5 \end{bmatrix}$

$$= \begin{bmatrix} -1 & 4 & 1 \\ 2 & 4 & -3 \end{bmatrix} + \begin{bmatrix} 6 & -6 & 6 \\ 12 & 0 & 30 \end{bmatrix} = \begin{bmatrix} 5 & -2 & 7 \\ 14 & 4 & 27 \end{bmatrix}.$$

[2]

(b) $\mathbf{CB} - 2\mathbf{DI}_2$ $= \begin{bmatrix} 1 & -1 & 1 \\ 2 & 0 & 5 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ 1 & 1 \\ -3 & -2 \end{bmatrix} - 2 \begin{bmatrix} 3 & -2 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} -2 & -6 \\ -11 & -16 \end{bmatrix} - \begin{bmatrix} 6 & -4 \\ 2 & 8 \end{bmatrix}$$

$$= \begin{bmatrix} -8 & -2 \\ -13 & -24 \end{bmatrix}.$$

[3]

(c) $\mathbf{AC} - \mathbf{BD}$

not defined : $\underset{2\times 3}{A} \; \underset{2\times 3}{C}$, $3 \neq 2$

[2]

(d) Suppose there is some matrix **P** such that the product **DCPA** is defined. What must be the size of the matrix **P**?

$$\underset{2\times 2}{D} \; \underset{2\times 3}{C} \; \underset{3\times 2}{P} \; \underset{2\times 3}{A}$$

$2 = 2$ ✓ must be equal

$\therefore P$ is 3×2

[3]

Question 3 [10 points]

(a) Determine $\mathbf{A}^{-1}$ where $\mathbf{A}$ is the matrix

$$\begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ -6 & 2 & 3 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 \\ -6 & 2 & 3 & 0 & 0 & 1 \end{array}\right]$$

$R_2 = (-1)r_1 + r_2$:
$R_3 = 6r_1 + r_3$:

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & -4 & 3 & 6 & 0 & 1 \end{array}\right]$$

$R_1 = 1 \cdot r_2 + r_1$:
$R_3 = 4r_2 + r_3$:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & -1 & 2 & 4 & 1 \end{array}\right]$$

$R_3 = (-1)r_3$:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -2 & -4 & -1 \end{array}\right]$$

$R_1 = 1 \cdot r_3 + r_1$:
$R_2 = 1 \cdot r_3 + r_2$:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & -3 & -1 \\ 0 & 1 & 0 & -3 & -3 & -1 \\ 0 & 0 & 1 & -2 & -4 & -1 \end{array}\right]$$

$$\therefore A^{-1} = \begin{bmatrix} -2 & -3 & -1 \\ -3 & -3 & -1 \\ -2 & -4 & -1 \end{bmatrix}$$

[5]

(b) Let $\mathbf{B} = \begin{bmatrix} 3 & 2 \\ 6 & 4 \end{bmatrix}$. Find $\mathbf{B}^{-1}$ or show that it does not exist.

$$\left[\begin{array}{cc|cc} 3 & 2 & 1 & 0 \\ 6 & 4 & 0 & 1 \end{array}\right]$$

$R_1 = \frac{1}{3}r_1$:

$$\left[\begin{array}{cc|cc} 1 & \frac{2}{3} & \frac{1}{3} & 0 \\ 6 & 4 & 0 & 1 \end{array}\right]$$

$R_2 = -6r_1 + r_2$:

$$\left[\begin{array}{cc|cc} 1 & \frac{2}{3} & \frac{1}{3} & 0 \\ 0 & 0 & -2 & 1 \end{array}\right]$$

does not reduce to I_2, so B^{-1} does not exist.

[2]

(c) Suppose a matrix $\mathbf{C}$ has inverse $\mathbf{C}^{-1} = \begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix}$. Use this to find a matrix $\mathbf{D}$ so that

$$\mathbf{CD} - \mathbf{I}_2 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$\Rightarrow CD = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + I_2 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 3 & 5 \end{bmatrix}$$

$$\therefore D = C^{-1}\begin{bmatrix} 2 & 2 \\ 3 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix}\begin{bmatrix} 2 & 2 \\ 3 & 5 \end{bmatrix} = \begin{bmatrix} 8 & 12 \\ 27 & 41 \end{bmatrix}.$$

[3]

Question 4 [10 points]: Maximize $z = 3x + 4y$ subject to the constraints

$$\begin{aligned} x + 2y &\geq 2 \\ 3x + 2y &\leq 12 \\ y &\leq 5 \\ x &\geq 0 \\ y &\geq 0 \end{aligned}$$

Keep your work organized: neatly draw any required graphs (use graph paper on the next page) and clearly show your work when determining corner points. State a clear conclusion.

Inequality	Line	Test Pt.	Test Result
$x+2y \geq 2$	$x+2y=2$	$(0,0)$	$0+2(0) \geq 2$: F
$3x+2y \leq 12$	$3x+2y=12$	$(0,0)$	$3(0)+2(0) \leq 12$: T
$y \leq 5$	$y=5$	$(0,0)$	$0 \leq 5$: T

Corner Points: • By inspection: $(4,0), (2,0), (0,1), (0,5)$.

• Solving: $\left.\begin{aligned} 3x+2y &= 12 \\ y &= 5 \end{aligned}\right\}$

$\therefore 3x + 2(5) = 12$

$3x = 2$

$x = \frac{2}{3}$

$\therefore \left(\frac{2}{3}, 5\right)$.

Corner Points	$z = 3x + 4y$
$(4,0)$	12
$(2,0)$	6
$(0,1)$	4
$(0,5)$	20
$(\frac{2}{3}, 5)$	22.

$\boxed{\therefore z \text{ has a maximum of } 22.}$

[10]

Question 4 (continued):