## **Elementary Row Operations**

The following **Elementary Row Operations (EROs)** change the form of a matrix without changing the solution of the corresponding system of equations:

- 1. Interchange any two rows [notation example:  $r_1 \leftrightarrow r_2$ ]
- 2. Multiply a row through by a non-zero constant [notation example:  $R_3 = (-2)r_3$ ]
- 3. Add a constant of one row to another row [notation example:  $R_1 = (-3)r_2 + r_1$ ]

### **Row Echelon Form**

A matrix is said to be in Row Echelon Form (REF) if

- 1. The first non-zero entry in any row is a 1 (called a **leading 1**).
- 2. The leading 1 in any row is located to the right of the leading 1 of any row above.
- 3. Any rows consisting entirely of zeros are at the bottom of the matrix.

#### Example:

| [ 1 | -4 | 2 | -1 | 3  |
|-----|----|---|----|----|
| 0   | 0  | 1 | 3  | -2 |
| 0   | 0  | 0 | 1  | 4  |
| 0   | 0  | 0 | 0  | 1  |
|     |    |   |    |    |
|     |    |   |    |    |

is in REF, but

| 1 | -4 | 2 | -1 | 3  |
|---|----|---|----|----|
| 0 | 0  | 0 | 1  | 4  |
| 0 | 0  | 1 | 3  | -2 |
| 0 | 0  | 0 | 0  | 1  |

is not (why?).

# Reduced Row Echelon Form

A matrix is said to be in **Reduced Row Echelon Form (RREF)** if, in addition to being in REF,

4. Any column containing a leading 1 has zeros elsewhere in the column.

#### Example:

| [1 | 1 | 0 | 5 |
|----|---|---|---|
| 0  | 0 | 1 | 3 |
| 0  | 0 | 0 | 0 |

is in RREF, but

| Γ0          | 1 | 0 | 5 |
|-------------|---|---|---|
| 0           | 0 | 1 | 3 |
| $\lfloor 1$ | 0 | 0 | 0 |

is not (why?). Nor is

| [1 | 1 | 0 | 5 |
|----|---|---|---|
| 0  | 0 | 1 | 3 |
| Γo | 0 | 1 | 0 |

# Gaussian Elimination Algorithm

To put a matrix into REF:

- 1. Interchange rows (if necessary) so that the first non-zero entry in the top row is located as far to the left as possible.
- 2. Use EROs to reduce the first entry in the top row to a leading 1. This can always be done by multiplying the top row by the reciprocal of the first entry in that row. A constant multiple of some other row can also be added to the top row to achieve this. Avoid introducing fractions if possible.
- 3. Now add suitable multiples of the top row to the rows below so that all entries below the leading 1 become zero.
- 4. Now, without changing or using the top row, go to step 1 and apply the procedure to the submatrix consisting of all rows below that containing the most recently used leading 1.
- 5. Proceed until there are no more rows.

# Gauss-Jordan Elimination Algorithm

To put a matrix into RREF, first put it into REF using Gaussian elimination procedure above, then perform an extra step:

6. Beginning with the last non-zero row and working upward, add suitable multiples of each row to the rows above to introduce zeros above the leading 1's.

## Interpreting the Outcome of the Algorithm

Each variable associated with a leading 1 in the REF (or RREF) is called a **leading variable**. Variables not associated with leading 1s are called **free variables**.

Gaussian (or Gauss-Jordan) elimination will result in exactly one of the following three outcomes:

- Some row of the REF (or RREF) will have zeros in all but the right-most position. In this case the system is **inconsistent** (has no solution.) Otherwise...
- The REF (or RREF) will have fewer non-zero rows than variables. In this case the system is **consistent** with infinitely many solutions. Solve for the leading variables in terms of the free variables which are considered **parameters** of the solution. Otherwise...
- The REF (or RREF) will have the same number of non-zero rows as variables and there is a single solution to the system. The system is **consistent** (meaning it has at least one solution).