

## Real Vector Spaces: Overview

Consider  $R^3 = \{(v_1, v_2, v_3) \mid v_i \in \mathbb{R}\}$ .

- We saw that  $R^3$  consists of all **linear combinations**

$$a \hat{\mathbf{i}} + b \hat{\mathbf{j}} + c \hat{\mathbf{k}}$$

where  $a$ ,  $b$  and  $c$  are scalars.

- $R^3$  is an example of a **vector space** over  $\mathbb{R}$ . We say that  $R^3$  is **spanned** by  $\{\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}\}$ .

- For any non-zero vector  $\mathbf{u}$  we saw that

$$W_1 = \{a \mathbf{u} \mid a \in \mathbb{R}\}$$

is a line through  $\mathbf{0}$ . Sums and scalar multiples of vectors in  $W_1$  are again in  $W_1$ .

- For any non-zero vectors  $\mathbf{u}$  and  $\mathbf{w}$  we saw that

$$W_2 = \{a \mathbf{u} + b \mathbf{w} \mid a, b \in \mathbb{R}\}$$

is a plane through  $\mathbf{0}$ . Sums and scalar multiples of vectors in  $W_2$  are again in  $W_2$ .

- $W_1$  and  $W_2$  are called **subspaces** of  $R^3$ .

These ideas are useful for studying other mathematical objects which can also be viewed as vectors.

## Definition of a Vector Space

Let  $V$  be a nonempty set of objects (called vectors) on which two operations are defined: addition and scalar multiplication.

**Addition** assigns to each pair of vectors  $\mathbf{u}$  and  $\mathbf{v}$  a vector denoted  $\mathbf{u} + \mathbf{v}$ .

**Scalar multiplication** assigns to each scalar  $k$  and vector  $\mathbf{u}$  a vector denoted  $k\mathbf{u}$ .

If for all vectors  $\mathbf{u}$ ,  $\mathbf{v}$ ,  $\mathbf{w}$  and scalars  $k$  and  $m$  the following **axioms** are satisfied then we say that  $V$  is a **vector space**:

1.  $\mathbf{u} + \mathbf{v}$  is in  $V$ . (Say that  $V$  is **closed** under addition.)
2.  $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
3.  $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
4. There is an object  $\mathbf{0}$  in  $V$  called the zero object such that  $\mathbf{0} + \mathbf{u} = \mathbf{u} + \mathbf{0}$  for every  $u$  in  $V$ .
5. For each  $\mathbf{u}$  in  $V$  there is  $-\mathbf{u}$  such that  $\mathbf{u} + (-\mathbf{u}) = (-\mathbf{u}) + \mathbf{u} = \mathbf{0}$ .
6.  $k\mathbf{u}$  is in  $V$  for all scalars  $k$ . (Say that  $V$  is **closed** under scalar multiplication.)
7.  $k(\mathbf{u} + \mathbf{v}) = k\mathbf{u} + k\mathbf{v}$
8.  $(k + m)\mathbf{u} = k\mathbf{u} + m\mathbf{u}$
9.  $k(m\mathbf{u}) = (km)\mathbf{u}$
10.  $1\mathbf{u} = \mathbf{u}$

For us the set of scalars will be the real numbers, in which case  $V$  is called a **real vector space**.

## Examples of Vector Spaces

1.  $\mathbb{R}^n$  with the usual addition and scalar multiplication of vectors.
2.  $\mathbb{R}^\infty$  = the set of all infinite sequences of real numbers  $\mathbf{u} = (u_1, u_2, u_3, \dots)$  with addition and scalar multiplication defined component-wise.
3.  $M_{mn}$  = the set of all  $m \times n$  matrices with matrix addition and scalar multiplication.

4.  $F(-\infty, \infty)$  = the set of all real valued functions with domain  $(-\infty, \infty)$ :

Suppose  $f, g$  are functions with domain  $(-\infty, \infty)$  and  $a, b$  are real scalars. Then  $af + bg$  defined by

$$(af + bg)(x) = af(x) + bg(x)$$

again has domain  $(-\infty, \infty)$ .

Here the vectors are functions.

5.  $C(-\infty, \infty)$  = the set of all real valued continuous functions with domain  $(-\infty, \infty)$ .

This time if  $f, g$  are continuous functions with domain  $(-\infty, \infty)$  and  $a, b$  are real scalars, then  $af + bg$  is again continuous with domain  $(-\infty, \infty)$ .

6.  $C^m(-\infty, \infty)$  = the set of all real valued functions whose first  $m$  derivatives exist and are continuous on  $(-\infty, \infty)$ .

7.  $P_n$  = the set of all polynomials of degree less than or equal to  $n$ :

$$p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$$

where  $a_0, a_1, a_2, \dots, a_n$  are real numbers.