Question 1: Consider the plane in R^3 which contains the points $P_1(-2,1,1)$, $P_2(0,-3,-2)$ and $P_3(1,2,3)$.

(a) Find a normal vector **n** to the plane.

$$\vec{n} = (\overrightarrow{P_1P_2}) \times (\overrightarrow{P_1P_3}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -4 & -3 \\ 3 & 1 & 2 \end{vmatrix} = \boxed{(-5,-13,14)}$$

[3]

(b) Find the area of the triangle in the plane which has vertices P_1, P_2 and P_3 .

$$A = \frac{1}{2}\|\vec{n}\| \quad \text{using (a)}$$

$$= \frac{1}{2}\sqrt{(-5)^2+(-13)^2+(14)^2}$$

$$= \boxed{\frac{\sqrt{390}}{2}}$$

[2]

(c) For the triangle with vertices P_1, P_2 and P_3, what is the sine of $\angle_{P_2P_1P_3}$, the angle at vertex P_1? ~~(Give your answer in degrees rounded to one decimal.)~~

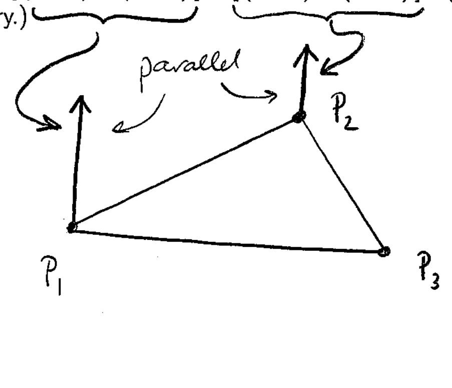
$$\sin\theta = \frac{\|(\overrightarrow{P_1P_2})\times(\overrightarrow{P_1P_3})\|}{\|\overrightarrow{P_1P_2}\|\ \|\overrightarrow{P_1P_3}\|} = \frac{\|\vec{n}\|}{\|(2,-4,-3)\|\ \|(3,1,2)\|}$$

$$= \frac{\sqrt{(-5)^2+(-13)^2+(14)^2}}{\sqrt{2^2+(-4)^2+(-3)^2}\sqrt{3^2+1^2+2^2}}$$

$$= \sqrt{\frac{390}{406}} \doteq \boxed{0.98}$$

[3]

(d) What is $[(\overrightarrow{P_1P_2}) \times (\overrightarrow{P_1P_3})] \times [(\overrightarrow{P_2P_1}) \times (\overrightarrow{P_2P_3})]$? (Don't work out the cross-products; think about the geometry.)



$\therefore$ cross-product is $\vec{0}$.

[2]

Question 2: Suppose that $\mathbf{A}$ is an $m \times n$ matrix. Show that the set of all solutions $\mathbf{x}$ to

$$\mathbf{A}\mathbf{x} = 0 \quad \Big\} \ *$$

is a subspace of R^n.

Let $\vec{x}_1$ and $\vec{x}_2$ be solutions to $*$,
and suppose that a, b are scalars.
We must show that $a\vec{x}_1 + b\vec{x}_2$ is also a solution to $*$:

$$\begin{aligned} A\left(a\vec{x}_1 + b\vec{x}_2\right) &= a(A\vec{x}_1) + b(A\vec{x}_2) \\ &= a(\vec{0}) + b(\vec{0}) \\ &= \vec{0}. \end{aligned}$$

[4]

Question 3: Recall that P_2 is the vector space of all polynomials of degree 2. Let $\mathbf{q}_1 = 2 + x + 4x^2$, $\mathbf{q}_2 = 1 - x + 3x^2$ and $\mathbf{q}_3 = 3 + 2x + 5x^2$. Determine if $\text{span}(\{\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3\}) = P_2$.

Let $\vec{p}$ be any polynomial in P_2.

$$a\vec{q}_1 + b\vec{q}_2 + c\vec{q}_3 = \vec{p}$$

$$\Rightarrow \quad \begin{bmatrix} 2 & 1 & 3 \\ 1 & -1 & 2 \\ 4 & 3 & 5 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \vec{p} \quad \Big\} \ *$$

$$\begin{vmatrix} 2 & 1 & 3 \\ 1 & -1 & 2 \\ 4 & 3 & 5 \end{vmatrix} = 2(-11) - 1(-3) + 3(7) = 2 \neq 0,$$

and so $*$ has solutions.

Since $\vec{p}$ was arbitrary, this shows that $\{\vec{p}_1, \vec{p}_2, \vec{p}_3\}$ spans P_2.

[4]

Question 4: Let $f_1(x) = x$, $f_2(x) = e^{-x}$ and $f_3(x) = e^{2x}$. Compute the Wronskian of f_1, f_2, f_3 and use it to show that the functions are linearly independent.

$$W = \begin{vmatrix} x & e^{-x} & e^{2x} \\ 1 & -e^{-x} & 2e^{2x} \\ 0 & e^{-x} & 4e^{2x} \end{vmatrix}$$

$$= x\left(-4e^{x} - 2e^{x}\right) - 1\left(4e^{x} - e^{x}\right) \quad \text{using col. 1}$$

$$= -6xe^{x} - 3e^{x}$$

$$\neq 0 \quad \text{for} \quad x = 0 \quad (\text{say}).$$

$\therefore$ f_1, f_2, f_3 are linearly independent.

[4]

Question 5: Let $\mathbf{u} = (1, a, 0)$, $\mathbf{v} = (a, 1, a)$ and $\mathbf{w} = (2, -2, a)$. Determine all values of a for which $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is a basis for R^3.

Find all a for which $\det\left[\vec{u} \mid \vec{v} \mid \vec{w}\right] \neq 0$.

$$\begin{vmatrix} 1 & a & 2 \\ a & 1 & -2 \\ 0 & a & a \end{vmatrix} = 1\,(3a) - a\,(a^2 - 2a) \quad \text{using col. 1}$$

$$= 3a - a^3 + 2a^2$$

$$= -a\,(a^2 - 2a - 3)$$

$$= -a\,(a-3)(a+1)$$

$\therefore$ $\det\left[\vec{u} \mid \vec{v} \mid \vec{w}\right] \neq 0$ for $a \neq 0, 3, -1$

[4]

Question 6: Determine the dimension of the vector space of all $n \times n$ diagonal matrices. Explain your answer.

Dimension is n since a basis is

$$S = \left\{ \underbrace{\begin{bmatrix} 1 & & & \\ & 0 & & \\ & & 0 & \\ & & & \ddots \; 0 \end{bmatrix}}_{n \times n}, \underbrace{\begin{bmatrix} 0 & & & \\ & 1 & & \\ & & 0 & \\ & & & \ddots \; 0 \end{bmatrix}}_{n \times n}, \ldots, \underbrace{\begin{bmatrix} 0 & & & \\ & 0 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}}_{n \times n} \right\}.$$

[2]

Question 7: Find a basis and state the dimension of the solution space of the following system:

$$\begin{aligned} x+y+z&=0\\ 3x+2y-2z&=0\\ 4x+3y-z&=0\\ 6x+5y+z&=0 \end{aligned}$$

$$\begin{bmatrix} 1 & 1 & 1 & 0\\ 3 & 2 & -2 & 0\\ 4 & 3 & -1 & 0\\ 6 & 5 & 1 & 0 \end{bmatrix}$$

$R_2=(-3)r_1+r_2$:
$R_3=(-4)r_1+r_3$:
$R_4=(-6)r_1+r_4$:

$$\begin{bmatrix} 1 & 1 & 1 & 0\\ 0 & -1 & -5 & 0\\ 0 & -1 & -5 & 0\\ 0 & -1 & -5 & 0 \end{bmatrix}$$

$R_2=(-1)r_2$:

$$\begin{bmatrix} 1 & 1 & 1 & 0\\ 0 & 1 & 5 & 0\\ 0 & -1 & -5 & 0\\ 0 & -1 & -5 & 0 \end{bmatrix}$$

$R_3=r_2+r_3$:
$R_4=r_2+r_4$:

$$\begin{bmatrix} 1 & 1 & 1 & 0\\ 0 & 1 & 5 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 \end{bmatrix}$$

let $z=t$, so $y=-5t$, $x=-t+5t=4t$

$\therefore$ solution is $\{t(4,-5,1) \mid t\in\mathbb{R}\}$,

so $\{(4,-5,1)\}$ is a basis for the solution space which has dimension 1.

[5]

Question 8: Let $\mathbf{v}_1=(1,-2,0,3)$, $\mathbf{v}_2=(2,-4,0,6)$, $\mathbf{v}_3=(-1,1,2,0)$ and $\mathbf{v}_4=(0,-1,2,3)$. Find a basis for the subspace of R^4 spanned by $\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3,\mathbf{v}_4\}$.

Equivalently, find a basis for the row space of

$$\begin{bmatrix} \vec{v}_1\\ \vec{v}_2\\ \vec{v}_3\\ \vec{v}_4 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 0 & 3\\ 2 & -4 & 0 & 6\\ -1 & 1 & 2 & 0\\ 0 & -1 & 2 & 3 \end{bmatrix}$$

$R_2=(-2)r_1+r_2$:
$R_3=r_1+r_3$:

$$\begin{bmatrix} 1 & -2 & 0 & 3\\ 0 & 0 & 0 & 0\\ 0 & -1 & 2 & 3\\ 0 & -1 & 2 & 3 \end{bmatrix}$$

$r_2 \leftrightarrow r_4$:

$$\begin{bmatrix} 1 & -2 & 0 & 3\\ 0 & -1 & 2 & 3\\ 0 & -1 & 2 & 3\\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_2=(-1)r_2$:

$$\begin{bmatrix} 1 & -2 & 0 & 3\\ 0 & 1 & -2 & -3\\ 0 & -1 & 2 & 3\\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_3=r_2+r_3$:

$$\begin{bmatrix} 1 & -2 & 0 & 3\\ 0 & 1 & -2 & -3\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ Basis is

$S=\{(1,-2,0,3),(0,1,-2,-3)\}$

[5]

Question 9: For this problem use the matrix $\mathbf{A} = \begin{bmatrix} 1 & -2 & 0 & 0 & 3 \\ 0 & 1 & 3 & 2 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

(i) What is rank($\mathbf{A}$) ?

3

[1]

(ii) What is nullity($\mathbf{A}$) ?

2

[1]

(iii) What is a basis for the column space of $\mathbf{A}$?

$\{(1,0,0,0), (-2,1,0,0), (0,3,1,0)\}$

[1]

(iv) What is a basis for the null space of $\mathbf{A}$?

$x_4 = r,\ x_5 = t \Rightarrow x_3 = -r,\ x_2 = -2r+3r = r,$
$x_1 = -3t+2r$

$\therefore$ Solutions are $\vec{x} = (-3t+2r, r, -r, r, t)$
$= r(2,1,-1,1,0) + t(-3,0,0,0,1)$

[2]

$\therefore$ basis is $S = \{(2,1,-1,1,0), (-3,0,0,0,1)\}$.

Question 10: Let W be the set of all polynomials $p(x)$ of degree 2 for which $p(1) = 0$. W is a subspace of P_2 and has dimension 2. Find a basis for W. Explain your solution. (This was a slightly more challenging homework problem.)

Consider $\vec{p}_1 = x-1$, $\vec{p}_2 = (x-1)^2$.
These are linearly independent (neither vector is a constant multiple of the other.)
Since we know that $\dim(W) = 2$, $\{\vec{p}_1, \vec{p}_2\}$ is then a basis.

[5]