Question 1: Compute the determinants of the following matrices. Show work or provide explanation to support your answers.

(a) $\mathbf{A} = \begin{bmatrix} 3 & 2 & 0 & -1 \\ 0 & 6 & 0 & 0 \\ 4 & 1 & 2 & 1 \\ 3 & -1 & 0 & 1 \end{bmatrix}$

using row 2: $\det(A) = 6 \begin{vmatrix} 3 & 0 & -1 \\ 4 & 2 & 1 \\ 3 & 0 & 1 \end{vmatrix}$

using col. 2

$= (6)(2)\begin{vmatrix} 3 & -1 \\ 3 & 1 \end{vmatrix}$

$= (6)(2)(6)$

$= \boxed{72}$

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(b) $\mathbf{B} = \begin{bmatrix} 4 & 2 & 1 & -1 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & -1 & 3 \\ 0 & 0 & 0 & 4 \end{bmatrix}$ upper triangular, so

$\det(B) = (4)(2)(-1)(4) = \boxed{-32}$

[2]

(c) $\mathbf{C} = \begin{bmatrix} 1 & 10 & 7 & -9 \\ 7 & -7 & 7 & 7 \\ 2 & -2 & 6 & 2 \\ -3 & -3 & 4 & 1 \end{bmatrix}$

$r_1 \leftrightarrow r_3$: $\det(C) = -\begin{vmatrix} 2 & -2 & 6 & 2 \\ 7 & -7 & 7 & 7 \\ 1 & 10 & 7 & -9 \\ -3 & -3 & 4 & 1 \end{vmatrix}$

$R_1 = \frac{1}{2} r_1$: $\det(C) = -2\begin{vmatrix} 1 & -1 & 3 & 1 \\ 7 & -7 & 7 & 7 \\ 1 & 10 & 7 & -9 \\ -3 & -3 & 4 & 1 \end{vmatrix}$

$R_2 = (-7)r_1 + r_2$:
$R_3 = (-1)r_1 + r_3$:
$R_4 = (3)r_1 + r_4$: $\det(C) = -2\begin{vmatrix} 1 & -1 & 3 & 1 \\ 0 & 0 & -14 & 0 \\ 0 & 11 & 4 & -10 \\ 0 & -6 & 13 & 4 \end{vmatrix}$

$= (-2)(14)\begin{vmatrix} 1 & -1 & 1 \\ 0 & 11 & -10 \\ 0 & -6 & 4 \end{vmatrix}$

$= (-2)(14)\left[44 - 60\right]$

$= \boxed{448}$

[5]

Question 2: Suppose that $\begin{vmatrix} -1 & -4 & 1 & 2 \\ 2 & -1 & -5 & 3 \\ -1 & 1 & 1 & 3 \\ 1 & -2 & -4 & 1 \end{vmatrix} = 69$. $= \det(A)$ say.

Use this fact and Cramer's rule to determine the value of x_3 in the following system of equations:

$$\begin{aligned} -x_1 - 4x_2 + x_3 + 2x_4 &= 0 \\ 2x_1 - x_2 - 5x_3 + 3x_4 &= 0 \\ -x_1 + x_2 + x_3 + 3x_4 &= 2 \\ x_1 - 2x_2 - 4x_3 + x_4 &= 0 \end{aligned}$$

$$A_3 = \begin{bmatrix} -1 & -4 & 0 & 2 \\ 2 & -1 & 0 & 3 \\ -1 & 1 & 2 & 3 \\ 1 & -2 & 0 & 1 \end{bmatrix}$$

$$\det(A_3) = 2\begin{vmatrix} -1 & -4 & 2 \\ 2 & -1 & 3 \\ 1 & -2 & 1 \end{vmatrix} = 2\left[(-1)(5) + 4(-1) + 2(-3)\right] = -30.$$

$$\therefore x_3 = \frac{\det(A_3)}{\det(A)} = \frac{-30}{69} = \boxed{\frac{-10}{23}}.$$

[6]

Question 3: Let $\mathbf{A} = \begin{bmatrix} 1 & a & -1 \\ a & 1 & 1 \\ 0 & 1 & a \end{bmatrix}$.

Determine all values of a for which $\mathbf{A}$ is non-singular (that is, for which $\mathbf{A}^{-1}$ exists.)

A non-singular $\Leftrightarrow \det(A) \neq 0$

$\det(A) = (-1)\begin{vmatrix} 1 & -1 \\ a & 1 \end{vmatrix} + a\begin{vmatrix} 1 & a \\ a & 1 \end{vmatrix}$ using row 3

$$\begin{aligned} &= -(1+a) + a(1-a^2) \\ &= -1 \cancel{-a} \cancel{+a} - a^3 \\ &= -(a^3+1) \end{aligned}$$

$\det(A) = 0 \Leftrightarrow a = -1$, so A is non-singular for $\boxed{a \neq -1}$

[4]

Question 4: Suppose **A** is an $n \times n$ matrix with the property that the sum of the entries in each row is zero. Show that **A** must have determinant zero. (Hint: consider the product **Ax** where **x** is an $n \times 1$ vector with all entries equal to 1.)

$A\vec{x} = \vec{0}$ since the sum of entries of each row of A is zero.

If $\det(A) \neq 0$ then A^{-1} exists,

and we would have $A^{-1}(A\vec{x}) = A^{-1}\vec{0}$,

giving $\vec{x} = \vec{0}$.

But $\vec{x} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$.

$\therefore$ $\det(A)$ must be zero.

[5]

Question 5: Let $\mathbf{u} = (3, -5, 1)$, $\mathbf{v} = (-2, 0, 2)$ and $\mathbf{w} = (0, -1, 4)$.

(i) Determine the distance between $-2\mathbf{u}$ and $\mathbf{v} + 3\mathbf{w}$.

$$\| -2\vec{u} - (\vec{v} + 3\vec{w}) \| = \| (-6, 10, -2) - (-2, -3, 14) \|$$

$$= \| (-4, 13, -16) \|$$

$$= \sqrt{(-4)^2 + (13)^2 + (-16)^2}$$

$$= \boxed{21}$$

[2]

(ii) Determine $\text{proj}_{\mathbf{w}}\mathbf{u}$.

$$\text{proj}_{\vec{w}}\vec{u} = \left(\vec{u} \cdot \frac{\vec{w}}{\|\vec{w}\|}\right)\left(\frac{\vec{w}}{\|\vec{w}\|}\right)$$

$$= (3, -5, 1) \cdot \frac{(0, -1, 4)}{\sqrt{17}} \left(\frac{(0, -1, +4)}{\sqrt{17}}\right)$$

$$= \frac{9}{17}(0, -1, +4)$$

$$\boxed{= \left(0, \frac{-9}{17}, \frac{36}{17}\right)}$$

[3]

Question 6: True or false: if **u** is orthogonal to both **v** and **w**, then **u** is also orthogonal to $a\mathbf{v} + b\mathbf{w}$ for every choice of scalars a and b. If true, prove it. If false, explain why.

True: $\vec{u}\cdot(a\vec{v}+b\vec{w}) = \vec{u}\cdot(a\vec{v}) + \vec{u}\cdot(b\vec{w})$

$= a(\underbrace{\vec{u}\cdot\vec{v}}_{0}) + b(\underbrace{\vec{u}\cdot\vec{w}}_{0})$

$= 0$

[2]

Question 7: Consider the points $P(-3,1,0,6)$, $Q(0,5,1,-2)$ and $R(-4,1,4,0)$ in R^4. Determine the angle between $\overrightarrow{PQ}$ and $\overrightarrow{PR}$. (State your answer in degrees rounded to one decimal.)

Let $\vec{u} = \overrightarrow{PQ} = (3,4,1,-8)$

$\vec{v} = \overrightarrow{PR} = (-1,0,4,-6)$

$\theta = \cos^{-1}\left[\dfrac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\,\|\vec{v}\|}\right]$

$= \cos^{-1}\left[\dfrac{-3+0+4+48}{\sqrt{9+16+1+64}\,\sqrt{1+0+16+36}}\right]$

$\doteq \boxed{44.8^\circ}$

[4]

Question 8: Find an equation of the line in R^3 that passes through the points $P(-2,1,3)$ and $Q(0,3,5)$. You may state your answer in either vector or parametric form.

Let $\vec{u} = (-2,1,3)$, $\vec{v} = \overrightarrow{PQ} = (2,2,2)$

Line is then $\vec{x} = \vec{u} + t\vec{v} = (-2,1,3) + t(2,2,2)$

$= (-2+2t, 1+2t, 3+2t)$.

[4]

Question 9: Determine whether the following planes are perpendicular:

$$\underbrace{x - 2y + 3z = 4,}_{} \quad \underbrace{-2x + 5y + 4z = -1}_{}$$

normal $\vec{n}_1 = (1,-2,3)$ normal $\vec{n}_2 = (-2,5,4)$

$$\vec{n}_1 \cdot \vec{n}_2 = (1,-2,3)\cdot(-2,5,4)$$
$$= -2 - 10 + 12$$
$$= 0$$

$\therefore$ Planes are perpendicular.

[3]

Question 10: Determine the distance between the following parallel planes:

$$3x - 4y + z = 1, \quad 6x - 8y + 2z = 3$$

Point on $3x-4y+z=1$ is $P_1(0,0,1)$

Point on $6x-8y+2z=3$ is $P_2(0,0,\frac{3}{2})$.

Let $\vec{u} = \overrightarrow{P_1P_2} = (0,0,\frac{1}{2})$,

normal to $3x-4y+z=1$ is $\vec{n} = (3,-4,1)$.

$\therefore$ distance between planes is

$$\|\text{proj}_{\vec{n}}\vec{u}\| = \left\|\left(\vec{u}\cdot\frac{\vec{n}}{\|\vec{n}\|}\right)\left(\frac{\vec{n}}{\|\vec{n}\|}\right)\right\|$$

$$= \frac{(0,0,\frac{1}{2})\cdot(3,-4,1)}{\sqrt{3^2+(-4)^2+1^2}}$$

$$= \frac{(\frac{1}{2})}{\sqrt{26}} = \boxed{\frac{1}{2\sqrt{26}}}$$

[7]